ESSENTIALS OF HYDRAULICS
(PART I)

SOLOMON ALEMU
1992
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By

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CHAPTER 1

INTRODUCTION

1.1 Definition of a Fluid

Matter is recognized to exist in everyday life in three states: solid, liquid and gas. Liquids and gases are called fluids since they are characterized by their ability to flow. The existence of matter in these states is governed by the spacing between different molecules and the intermolecular attractive forces. The molecules in the solid state are spaced very closely and the intermolecular attractive forces are very strong thus imparting to solids the property of compactness and rigidity of form. On the other hand, as a result of weaker intermolecular attractive forces, liquid molecules can move more freely within the liquid mass and consequently liquids do not possess any rigidity of form but take the shape of the container in which they are kept. However, a definite mass of a liquid occupies a definite volume. The intermolecular forces are extremely weak in gases and the molecules are so farther apart spaced that gases do not have a definite volume like liquids and solids. Consequently a given mass of gas fills the container in which it is placed regardless of the size of the container. A liquid offers greater resistance to volumetric change (compression) and is not greatly affected by temperature changes. A gas, on the other hand, is easily compressible and responds markedly to temperature changes.

It is more appropriate to classify matter as fluids and solids on the basis of its response to the application of external forces. On this basis, a fluid may be defined as a substance which deforms continuously under the action of shear forces, however small these forces may be. This implies that if a fluid is at rest there can be no shearing forces acting. This property distinguishes fluids from solids. Solids acquire an equilibrium deformation corresponding to an internal stress
that develops to just balance the applied external stress. Liquids do not acquire an equilibrium distortion but continue to deform as long as the stress acts.

1.2. The subject Matter of Hydraulics:

The branch of mechanics which treats the equilibrium and motion of liquids and gases and the force interactions between them and the bodies through or around which they flow is called hydromechanics or fluid mechanics. Hydraulics is an applied division of fluid mechanics governing a specific range of engineering problems and methods of their solution.

The principal concern of hydraulics is the study of fluids at rest and fluid flow constrained by surrounding surfaces, i.e., flow in open and closed channels and conduits, including rivers, canals and flumes, as well as pipes, nozzles and hydraulic machines with internal flow of fluids. It investigates what might be called "internal" problems, as distinct from "external" problems involving the flow of continuous medium about submerged bodies as in the case of a solid body moving in water or in the air. These "external" problems are treated in hydrodynamics and aerodynamics in connection with ship and aircraft design.

The science of hydraulics concerns itself mainly with the motion of liquids. However, under certain conditions, the laws of motion of liquids and gasses are practically identical as for example in the study of internal flows of gases with velocities much lower than that of sound in which case their compressibility can be disregarded. Hydraulics provides the methods of designing a wide range of hydraulic structures (dams, canals weirs pipelines etc), machinery (pumps, turbines, fluid couplings) and other devices in many branches of engineering.
Fluid flow problems in hydraulics are investigated by first simplifying and idealizing the phenomenon under investigation and applying the laws of theoretical mechanics. The results are then compared with experimental data, discrepancies are established and the theoretical formulae and solutions adjusted so as to make them suitable for practical application. Some phenomena are so involved as to defy theoretical analysis and are investigated in hydraulics on the sole basis of experimental measurement. Thus, hydraulics can be called a semi-empirical science.

1.3 Dimensions and Units of Measurement:

Physical quantities such as displacement, velocity, force etc are represented by dimensions. A unit is a particular way of describing the magnitude of a dimension. Thus length is a dimension associated with variables such as distance, displacement, width, deflection and height while centimeters and inches are both units used to describe the magnitude of the dimension length.

The dimensions length \( \{L\} \), time \( \{T\} \), mass \( \{M\} \) and force \( \{F\} \) are of fundamental interest in fluid mechanics. The dimensions of other, derived, physical quantities may be established by applying the above dimensions to the definition of the physical quantity under consideration as follows:
From the four fundamental dimensions given earlier only three need be selected as basic since force and mass are related through Newton's Second Law of motion. Thus if one is chosen as a fundamental dimension in any consistent dimensional system the other becomes a derived dimension. According to the choice made, two systems of measurement result. These are the force (or gravitational) system in which the basic dimensions are Force, Length and Time and the Mass (or Absolute) system in which the basic dimensions are Mass, Length and Time.

Thus the FPS (British) system is a force (gravitational) system while the MKS (metric) system is a mass (Absolute) system of measurement.

**The S.I. System**

The Absolute Metric System of units in which kilogram is the unit of mass, meter is the unit of length and second is the unit of time, forms the basis of an internationally agreed system of units - the Systeme Internationale d' unites - designated SI, which is now being adopted by almost all countries.

<table>
<thead>
<tr>
<th>Physical quantity</th>
<th>Definition</th>
<th>Derived dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity</td>
<td>Displacment/Time</td>
<td>(\frac{L}{T} = LT^{-1})</td>
</tr>
<tr>
<td>Acceleration</td>
<td>Velocity/Time</td>
<td>(LT^{-1}/T = LT^{-2})</td>
</tr>
<tr>
<td>Force</td>
<td>Mass x Acceleration</td>
<td>(M.LT^{-2})</td>
</tr>
<tr>
<td>Mass</td>
<td>Force ÷ Acceleration</td>
<td>(F.L^{-1}T^{2})</td>
</tr>
</tbody>
</table>
The basic SI units are the following:

<table>
<thead>
<tr>
<th>Quantity</th>
<th>SI Units</th>
<th>Abbreviations in SI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>Meter</td>
<td>m</td>
</tr>
<tr>
<td>Mass</td>
<td>Kilogram</td>
<td>kg</td>
</tr>
<tr>
<td>Time</td>
<td>Second</td>
<td>s</td>
</tr>
<tr>
<td>Electric current</td>
<td>Ampere</td>
<td>A</td>
</tr>
<tr>
<td>Temperature</td>
<td>Kelvin</td>
<td>K</td>
</tr>
<tr>
<td>Luminous intensity</td>
<td>Candela</td>
<td>Cd.</td>
</tr>
<tr>
<td>Amount of substance</td>
<td>Mole</td>
<td>Mol.</td>
</tr>
<tr>
<td>Plane angle</td>
<td>radian</td>
<td>rad.</td>
</tr>
</tbody>
</table>

The following are the derived units of interest in fluid mechanics:

<table>
<thead>
<tr>
<th>Quantity</th>
<th>SI Units</th>
<th>Abbreviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Force</td>
<td>Newton</td>
<td>N {1N = 1kg \cdot m/s^2}</td>
</tr>
<tr>
<td>Pressure (stress)</td>
<td>Pascal</td>
<td>P_a {1P_a = 1N/m^2}</td>
</tr>
<tr>
<td>Work, energy, quantity of heat</td>
<td>Joule</td>
<td>J {1J = 1N \cdot m}</td>
</tr>
<tr>
<td>Power</td>
<td>Watt</td>
<td>W {1W = 1J/s}</td>
</tr>
<tr>
<td>Dynamic viscosity</td>
<td>Pascal-second</td>
<td>P_a \cdot s</td>
</tr>
<tr>
<td>Kinematic viscosity</td>
<td>Squaremetre per second</td>
<td>m^2/s</td>
</tr>
<tr>
<td>Surface Tension</td>
<td>Newtons per meter</td>
<td>N/m</td>
</tr>
<tr>
<td>Momentum</td>
<td>Kilogram x meter/second</td>
<td>Kg.m/s</td>
</tr>
</tbody>
</table>
Regarding the unit of force, 1N is the force required to give a 1 kg mass an acceleration of 1 m/s². Hence 1N = 1 kg m/s². Since W = mg, the weight or the force of gravity of 1 kg mass is 1 kg \( 9.806 \text{ m/s}^2 = 9.806 \text{ kg m/s}^2 = 9.806 \text{ N} \). Standard acceleration due to gravity is 9.806 m/s².

Abbreviations of SI units are written in small letters eg. hours (h), meters (m). When a unit is named after a person, the abbreviation (but not the spelled form) is capitalized, examples are watt (W), pascal (Pa), newton (N). Common prefixes are shown below:

<table>
<thead>
<tr>
<th>Multiple</th>
<th>SI Prefix</th>
<th>Abbreviation</th>
<th>Multiple</th>
<th>SI Prefix</th>
<th>Abbreviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10^9)</td>
<td>giga</td>
<td>G</td>
<td>(10^3)</td>
<td>milli</td>
<td>m</td>
</tr>
<tr>
<td>(10^6)</td>
<td>mega</td>
<td>M</td>
<td>(10^6)</td>
<td>micro</td>
<td>(\mu)</td>
</tr>
<tr>
<td>(10^3)</td>
<td>kilo</td>
<td>K</td>
<td>(10^9)</td>
<td>nano</td>
<td>n</td>
</tr>
<tr>
<td>(10^2)</td>
<td>centi</td>
<td>C</td>
<td>(10^{-12})</td>
<td>pico</td>
<td>p</td>
</tr>
</tbody>
</table>
CHAPTER 2
PROPERTIES OF FLUIDS

2.1. Introduction:

The properties of fluids vary from fluid to fluid and have a decisive influence on the motion of a fluid. Thus, it is not necessary to deal with each fluid separately while studying fluid motion. One needs to study only the variation of these properties and the manner in which they influence the fluid motion. This chapter discuss fluid properties and their significance.

2.2 Density, Specific Weight, Specific Volume and Specific Gravity

Density of a fluid, designated by the symbol \( \rho \) (Rho), is probably the most important property. It is defined as the fluid mass per unit volume. In the S.I. system density is expressed in kg/m\(^3\). Generally the density of a fluid is dependent on temperature and pressure. For water at 4\(^\circ\)C and standard pressure (i.e. 760 mm of mercury), \( \rho = 1000 \) kg/m\(^3\).

Specific Weight (or Unit Weight) is defined as the weight of fluid per unit volume. It is designated by \( \gamma \) (Gama). It could also be seen as representing the force exerted by gravity on a unit volume of fluid. The unit of specific weight in the SI system is N/m\(^3\). Density and specific weight may be related as follows:

\[
\text{Since } \gamma = \frac{W}{V} = \frac{mg}{V}, \text{ then } \gamma = \rho g.
\]

The specific weight of water at 4\(^\circ\)C is 9810 N/m\(^3\).
**Specific Volume V** is the volume of the fluid per unit weight. It is the reciprocal of specific weight so that $V = \frac{1}{\gamma}$ with units of $m^3/N$. It is a property commonly used in gas flow problems.

**Specific Gravity S** (also known as relative density) is the ratio of the mass of a fluid to the mass of an equal volume of pure water at standard temperature and pressure. It may also be defined as the density of the fluid to the density of pure water at standard conditions. As a ratio, specific gravity is dimensionless. The specific gravity of pure water is unity while that of mercury is about 13.60.

### 2.3 Pressure, Compressibility, Viscosity:

**Pressure:** The normal force exerted against a plane area divided by the area is the average pressure on the area. Fluids exert pressure on the walls and the bottom of containers in which they are stored. If $\Delta F$ is the force exerted over an area $\Delta A$, then the pressure $P$ is given by:

$$P = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

Pressure $P$ has the dimension force per unit area. In the SI system, the unit of $P$ is $N/m^2$ which is called pascal (Pa). Pressure is also expressed in bars, where 1 bar = 100,000 $N/m^2$.

**Compressibility:** All fluids may be compressed by the application of pressure, elastic energy being stored in the process. As a result of compressibility, fluid density changes with pressure. Gases are highly compressible and hence are treated as such. In liquids, the change in density (and therefore in volume) is very small even under large pressure changes. Therefore, liquids are ordinarily considered as incompressible. But in special problems such as Water Hammer involving sudden or great changes in pressure, the
compressibility of the liquid becomes important and should be taken into consideration.

The compressibility of a fluid is expressed by defining a modulus of elasticity as in done for solids. But since fluids do not possess rigidity of form, the modulus of elasticity must be defined on the basis of volume; such a modulus being termed Bulk Modulus of Elasticity $K$.

In order to define the Bulk Modulus of Elasticity, consider a compressible fluid in a cylinder of cross-sectional area $A$, which is being compressed by a piston as shown in Figure 2.1. The cylinder and the piston are considered rigid.

![Figure 2.1](image)

Let the original volume of the fluid be $V_0$. The application of a force $F$ results in the pressure $P = F/A$ exerted on the fluid. This pressure reduces the fluid volume to $V$. A plot of $V/V_0$ (which is a measure of volumetric strain) against the pressure $P$ results in a curve of negative slope as shown in Figure 1(b). The Bulk Modulus of Elasticity $K$ of the fluid corresponding to a pressure $P_1$ is defined as:
The negative sign indicates the decrease in $\frac{dv}{V_0}$ with increase in pressure. Since $\frac{dv}{V_0}$ is dimensionless, the dimension of $K$ is the same as that of the pressure $P$. Water has an average value of $K = 2.1$ GPa. This shows that water is about 100 times more compressible than steel, but it is ordinarily considered incompressible.

Table 2.1  Bulk Modulus of Elasticity of Water

<table>
<thead>
<tr>
<th>Pressure (MPa)</th>
<th>Temperature</th>
<th>0°C</th>
<th>10°C</th>
<th>20°C</th>
<th>50°C</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1 - 2.5</td>
<td>1.93</td>
<td>2.03</td>
<td>2.07</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.5 - 5.0</td>
<td>1.96</td>
<td>2.06</td>
<td>2.13</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.0 - 7.5</td>
<td>1.99</td>
<td>2.14</td>
<td>2.23</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.5 - 10.0</td>
<td>2.02</td>
<td>2.16</td>
<td>2.24</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.0 - 50.0</td>
<td>2.13</td>
<td>2.27</td>
<td>2.34</td>
<td>2.43</td>
<td></td>
</tr>
<tr>
<td>50.0 - 100.0</td>
<td>2.43</td>
<td>2.57</td>
<td>2.67</td>
<td>2.77</td>
<td></td>
</tr>
<tr>
<td>100 - 150</td>
<td>2.84</td>
<td>2.91</td>
<td>3.00</td>
<td>3.11</td>
<td></td>
</tr>
</tbody>
</table>

Example 2.1. What pressure increase is required to reduce the volume of 100 c.c of water by 0.5%? $K = 2.1$ GPa

Solution:

\[
K = -\frac{dP}{d(V/V_0)} = -\frac{dP}{dv/V_0} = \frac{100 \, dP}{0.5} = 200 \, dP
\]

Thus \[ dP = \frac{K}{200} = \frac{2.1 \times 10^9}{200} \, Pa = 1.05 \times 10^7 \, Pa \]
Since 1 atmosphere $= 1 \times 10^5$ Pa, the required increase in pressure is 105 atmospheres. This is an extremely high pressure which is required to produce a volume change of only 0.5%. Hence the reasonableness of the assumption that liquids are practically incompressible under ordinary changes in pressure.

Viscosity: Viscosity is one of the most important physical properties of fluids. It is a measure of the resistance of a fluid to relative motion such as shear and angular deformation within the fluid. Viscosity is due to interchange of molecules between adjacent layers of fluids moving at different velocities and also to the cohesion between fluid particles. Viscosity plays a decisive role in laminar flow and fluid motion near solid boundaries.

The relationship between viscous shear stress and viscosity is expressed by Newton's law of viscosity. Consider a fluid confined between two plates separated by a small distance $y$ as shown in Figure 2.2. The lower plate is stationary while the upper plate is moved with a velocity $v$.

Since there will not be slippage between the plates and the fluid, particles of fluid in contact with each plate will
adhere to it i.e. fluid particles in contact with the moving plate will have a velocity $V$ while those in contact with the stationary plate will have zero velocity. The effect is as if the fluid were made up of a series of thin, parallel layers each moving a little faster relative to the adjacent, lower layer.

For a large number of fluids, the shear stress developed between adjacent layers of fluid is found to be directly proportional to the rate of change of velocity with respect to $y$, which is the velocity gradient. For a layer of thickness $dy$ at a distance $y$ from the stationary plate this becomes:

$$\tau \propto \frac{du}{dy}$$

Introducing a constant of proportionality $\mu$, one obtains:

$$\tau = \mu \frac{du}{dy} \quad (2.2)$$

The proportionality constant $\mu$ expresses the property of the particular fluid and is called dynamic viscosity. Equation 2.2 is called Newton's Law of viscosity.

Fluids may be classified on the basis of the relationship between the shear stress $\tau$ and the rate of deformation (velocity gradient) as shown in Figure 2.3.

Fluids may be classified as Newtonian and non-Newtonian. In Newtonian fluids there is a linear relation between the magnitude of the applied shear stress $\tau$ and the resulting rate of deformation i.e. $\mu$ is constant. In non-Newtonian fluids there is a non-linear relation between the applied shear stress and the rate of angular deformation. An ideal plastic has a
definite yield stress and a constant linear relation between \( \tau \) and \( \frac{du}{dy} \) thereafter. A thixotropic substance, such as printer's ink, has a viscosity that is dependent upon the prior angular deformation of the substance and has a tendency of setting when at rest. Gases and thin liquids such as water, kerosene, glycerin etc are Newtonian fluids.

The viscosity of a fluid is a function of temperature. Since the viscosity of liquids is governed by cohesive forces between the molecules, it decreases with increase in temperature. In gases, however, molecular momentum transfer plays a dominant role in viscosity and as a result the viscosity of gases increases with increase in temperature.

The unit of dynamic viscosity \( \mu \) is Ns/m\(^2\) or kg/ms. A smaller unit of dynamic viscosity is called the poise. 1 poise = 1 gm/cm.s. Thus 1 poise = 0.1 kg/ms.

Kinematic viscosity \( (v) \) is the ratio of dynamic viscosity to density. \( v = \frac{\mu}{\rho} \) and has units of m\(^2\)/s. A smaller unit of
\( \nu \) is the stoke. 1 stoke = 1 cm\(^2\)/s. Thus 1 stoke = 1 \( \times 10^4 \) m\(^2\)/s.

Table 2.2 Dynamic and Kinematic Viscosities of Water and Carbon Tetrachloride.

<table>
<thead>
<tr>
<th>Temp ( ^\circ C )</th>
<th>Water</th>
<th></th>
<th></th>
<th>Carbon Tetra Chloride</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \rho ) (kg/m(^3))</td>
<td>( \mu ) (10(^4)P.s)</td>
<td>( \nu ) (10(^4)m(^2)/s)</td>
<td>( \rho ) (kg/m(^3))</td>
<td>( \mu ) (10(^4)P.s)</td>
<td>( \nu ) (10(^4)m(^2)/s)</td>
</tr>
<tr>
<td>0</td>
<td>999.8</td>
<td>17.53</td>
<td>1.75</td>
<td>1633</td>
<td>13.46</td>
<td>0.824</td>
</tr>
<tr>
<td>10</td>
<td>999.7</td>
<td>13.00</td>
<td>1.30</td>
<td>1613</td>
<td>11.34</td>
<td>0.703</td>
</tr>
<tr>
<td>20</td>
<td>998.2</td>
<td>10.02</td>
<td>1.004</td>
<td>1594</td>
<td>9.708</td>
<td>0.609</td>
</tr>
<tr>
<td>30</td>
<td>995.7</td>
<td>7.972</td>
<td>0.801</td>
<td>1575</td>
<td>8.418</td>
<td>0.534</td>
</tr>
<tr>
<td>40</td>
<td>992.2</td>
<td>6.514</td>
<td>0.657</td>
<td>1555</td>
<td>7.379</td>
<td>0.475</td>
</tr>
<tr>
<td>50</td>
<td>988.0</td>
<td>5.542</td>
<td>0.555</td>
<td>1535</td>
<td>6.529</td>
<td>0.425</td>
</tr>
</tbody>
</table>

2.4. Surface Tension, Capillarity and Vapour Pressure

These are strictly liquid properties.

*Surface Tension and Capillarity* are due to properties called cohesion and adhesion. Cohesion is the property as a result of which molecules of a liquid stick to each other whereas adhesion is the property that enables liquids to stick or adhere to another body. As a result of cohesion an imaginary film capable of resisting some tension is created at a free liquid surface. The liquid property that creates this capability is called surface tension. It is because of surface tension that a small pin placed gently on water surface will not sink but remain floating being supported by the tension at the water surface. The spherical shape of water drops is also due to surface tension. Surface tension force, designated by \( \sigma \), is defined as force per unit length and has the unit N/m.
For water in contact with air $\delta$ varies from about 0.074 N/m at 0°C to 0.059 N/m at 100°C. Surface tension force is so small that it is neglected in ordinary hydraulic problems. It is, however, a factor to be taken into consideration in flows at small depths that occur in model studies. Capillarity is due to both adhesion and cohesion. If a glass tube of small diameter and open at both ends is dipped in a container of water, the water rises in the tube to some height above the level of water in the container (Figure 2.4 a). If the same tube is dipped in a container of mercury, the level inside the tube will be lower than that in the container (Figure 2.4 b). In the former, adhesion of water to glass is predominant in comparison to the cohesion whereas in the latter cohesion between mercury molecules is predominant.

Capillary rise or Capillary drop can be estimated by considering the equilibrium of the liquid column of height $h$ as follows:

Consider a tube of small internal diameter $D = 2r$ dipped in a liquid of specific weight $\gamma$ and surface tension force $\sigma$. 

![Capillary Rise and Drop Diagram](image)
Let the liquid rise to a height \( h \) in the tube as a result of capillarity (Figure 2.5).

\[
2\pi r. \sigma. \cos \theta = \pi r^2 h. \gamma
\]

from which, \( h = \frac{2\sigma \cos \theta}{\gamma r} = \frac{4\sigma \cos \theta}{\gamma D} \) \hspace{1cm} (2.3)

This shows that the capillary rise is inversely proportional to the diameter of the tube. Hence for tubes of very small diameter, the capillary rise can be considerable. Therefore, to minimize the effect of capillarity, the tubes of manometers and piezometers should not be less that about 10 mm in diameter. If the surface is clean, the contact angle is zero for water and about 140° for mercury.
Table 2.3 Surface Tension of Water (N/m)

<table>
<thead>
<tr>
<th>Temperature °C</th>
<th>0°</th>
<th>10°</th>
<th>20°</th>
<th>30°</th>
<th>40°</th>
<th>50°</th>
<th>60°</th>
</tr>
</thead>
<tbody>
<tr>
<td>σ</td>
<td>0.0742</td>
<td>0.0728</td>
<td>0.0713</td>
<td>0.0698</td>
<td>0.0682</td>
<td>0.069</td>
<td>0.0661</td>
</tr>
</tbody>
</table>

Vapour Pressure: Liquids evaporate when their surface is exposed to the atmosphere. The nature of the liquid, its temperature and the prevailing atmospheric pressure determine the rate of evaporation. If a closed container is partially filled with a liquid maintained at a constant temperature, evaporation will take place and the vapor molecules accumulating in the space above the liquid exert a pressure called vapour pressure $P_v$ on the liquid surface. In the process of evaporation, some of the vapour molecules get reabsorbed into the liquid. With the passage of time an equilibrium situation is established whereby the number of molecules released from the liquid become equal to the number of molecules reabsorbed and the vapour pressure becomes constant. This vapour pressure is called the saturation vapour pressure. Increase of temperature hastens evaporation because of increased molecular activity and consequently saturation vapour pressure is increased with temperature. A liquid having high vapour pressure evaporates more easily than a liquid with low vapour pressure. Thus Carbon Tetra Chloride, with a saturation vapour pressure of $1.275 \times 10^4$ N/m$^2$ at 20°C, evaporates easily compared to Mercury, which has a saturation vapour pressure of only $0.17$ N/m$^2$ at 20°C. This is one of the reasons that make mercury an ideal liquid for barometers and manometers.

Boiling of a liquid will takes place, at any temperature, when the external absolute pressure impressed on a liquid surface is equal to or less than the saturation vapour pressure of the liquid. In liquid flow system, very low pressures may be produced at certain points in the system. If these
pressures are less than or equal to the vapour pressure of the liquid, the liquid flashes into vapour creating vapour pockets. These vapour pockets collapse as they are swept into regions of higher pressure. This phenomenon is called Cavitation and can result in damages of conduit walls and propeller runner blade tips where low pressures are likely to develop. Table 2.4 gives some values of saturation vapour pressure of water.

### Table 2.4 Saturation Vapour Pressure of Water

<table>
<thead>
<tr>
<th>Temperature $\circ C$</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_v (x10^2 \text{ N/m}^2)$</td>
<td>0.6108</td>
<td>1.227</td>
<td>2.337</td>
<td>4.242</td>
<td>7.377</td>
<td>12.33</td>
</tr>
</tbody>
</table>

**Example 2.2.** An oil has a density of 850 kg/m$^3$ at 20°C. Find its specific gravity and Kinematic viscosity if the dynamic viscosity is $6 \times 10^{-3}$ kg/ms.

**Solution:** Specific gravity of oil, $S_o = \frac{\rho_{\text{oil}}}{\rho_{\text{water}}}$

$$ = \frac{850}{1000} = 0.85$$

Kinematic viscosity, $v_o = \frac{\mu}{\rho} = 6 \times 10^{-3} / 850$

$$ = 7.06 \times 10^{-6} \text{ m}^2/\text{s}.$$  

**Example 2.3.** The velocity distribution over a plate in a fluid ($\mu = 8.63 \text{ Poise}$) is given by $u = \frac{2}{3}y - y^2$ where $u$ is the velocity in m/s at a distance $y$ metres above the plate. Determine the shear stress at the plate and at a distance of 0.15 metres from the plate.
Solution:

\[ u = \frac{2}{3} y - y^2 \]

Therefore \( \frac{du}{dy} = \frac{2}{3} - 2y \)

Hence \( \frac{du}{dy} = \frac{2}{3} \) at \( y = 0 \)

and \( \frac{du}{dy} = \frac{2}{3} - 0.30 = 0.367 \) at \( y = 0.15 \) m

\[ \mu = 8.63 \text{ poise} = 0.863 \text{ Ns/m}^2 \]

The shear stress \( \tau = \mu \frac{du}{dy} \)

Thus, at \( y = 0 \), \( \tau = 0.863 \times \frac{2}{3} = 0.575 \text{ N/m}^2 \)

at \( y = 0.15 \), \( \tau = 0.863 \times 0.367 \)

\[ = 0.317 \text{ N/m}^2 \]

Example 2.4. What should be the diameter of a droplet of water in mm at 20°C if the pressure inside is to be 170 N/m² greater than the outside?

Solution:

Let the diameter of the droplet be \( d \) and the internal pressure be \( p \). If the droplet is cut into two halves forces acting on one half will be those due to pressure intensity \( p \) on the area \( \frac{\pi d^2}{4} \)

and the force due to surface tension \( \sigma \) acting around the circumference \( \pi d \). These two force will be equal and opposite under equilibrium condition.

\[ p \times \frac{\pi d^2}{4} = \sigma \cdot \pi d \]

\[ \therefore d = \frac{4\sigma}{p} \]
at 20°C, \( \sigma = 0.074 \text{ N/m} \)

Therefore \( d = \frac{4 \times 0.074}{170} = 1.74 \times 10^{-3} \text{ m} = 1.74 \text{ mm} \)

Example 2.5. The density of a certain oil at 20°C is 800 kg/m³. Find its specific gravity and kinematic viscosity if the dynamic viscosity is \( 5 \times 10^{-3} \text{ kg/m.s} \).

Solution:

Specific gravity, \( s = \frac{\rho \text{ of oil}}{\rho \text{ of water}} \)
\[ = \frac{800}{10^3} = 0.80 \]

Kinematic viscosity, \( \nu = \frac{\mu}{\rho} \)
\[ = \frac{5 \times 10^{-3}}{800} = 6.3 \times 10^{-6} \text{ m}^2/\text{s} \]

Example 2.6. The velocity distribution of a viscous fluid over a fixed boundary is given by \( u = 0.72y - y^2 \) in which \( u \) is the velocity in m/s at a distance \( y \) metres above the boundary surface. If the dynamic viscosity \( \mu \) of the fluid is 0.92 Ns/m², determine the shear stress at the surface and at \( y = 0.36 \text{ m} \).

Solution:

\( u = 0.72y - y^2 \)
\[ \therefore \frac{du}{dy} = 0.72 - 2y \]
At the surface, \( y = 0 \) and \( \frac{du}{dy} = 0.72 \text{ s}^{-1} \)
At \( y = 0.36 \), \( \frac{du}{dy} = 0.72 - 2(0.36) = 0 \)
Since \( \tau = \mu \frac{du}{dy} \)
At the surface, \( \tau = 0.92 \times 0.72 = 0.662 \text{ N/m}^2 \)
At \( y = 0.36 \), \( \tau = 0.92 \times 0 = 0 \)
Example 2.7. At a depth of 6.8 Km in the ocean the pressure is 72 MN/m². The specific weight of sea water at the surface is 10.2 KN/m³ and its average bulk modulus is 2.4 x 10⁶ KN/m. Determine (a) The specific volume (b) The change in specific volume over the depth, and (c) the specific weight of sea water at 6.8 Km depth.

Solution:

change in pressure dp from surface to 6.8 Km depth

\[ \text{dp} = 72 \text{ MN/m}^2 \]

\[ = 7.2 \times 10^4 \text{ KN/m}^2 \]

Bulk modulus, \( k = \frac{-dp}{dv/V} \)

\[ \therefore \frac{dv}{V} = \frac{-dp}{K} = 7.2 \times 10^4 / 2.4 \times 10^6 = 3 \times 10^{-2} \]

a) Specific volume = volume per unit weight = 1/Y

\[ \therefore \text{specific volume at the surface} = 1/10.2 = 9.8 \times 10^{-2} \text{ m}^3/\text{KN}. \]

b) Change in specific volume between that at the surface and at 6.8 Km depth, \( dv = 3 \times 10^{-2} \times 9.8 \times 10^{-2} = 29.4 \times 10^{-4} \text{ m}^3/\text{kN} \)

c) The specific volume at 6.8 Km depth = 9.8 \times 10^{-2} - 29.4 \times 10^{-4} \]

\[ = 9.51 \times 10^{-2} \text{ m}^3/\text{kN} \]

\[ \therefore \text{The specific weight at 6.8 Km depth} = 1/\text{specific volume} \]

\[ = 1/9.51 \times 10^{-2} \]

\[ = 10.52 \text{ kN/m}^3 \]

Example 2.8. Calculate the capillary effect in mm in a glass tube of 6 mm diameter when immersed in (a) Water \( \sigma = 73 \times 10^{-3} \text{ N/m} \) and (b) Mercury, \( \sigma = 0.5 \text{ N/m} \). The contact angles for Water and Mercury are zero and 130° respectively. Take specific weights of water and mercury to be 9810 N/m³ and 1.334 \times 10^5 N/m³ respectively.
Solution:

Capillary rise (drop), \( h = \frac{4\sigma \cos \alpha}{\gamma \cdot d} \), where \( d \) is tube diameter.

For Mercury: Capillary drop,

\[
h = \frac{4 \times 0.5 \times \cos 130°}{1.334 \times 10^5 \times 6 \times 10^{-3}} = -1.61 \times 10^{-3} \text{ m}
\]

\[
= -1.61 \text{ mm} = 1.61 \text{ mm depression}
\]

For Water: Capillary rise

\[
h = \frac{4 \times 73 \times 10^{-3} \times \cos 0°}{9810 \times 6 \times 10^{-3}} = 4.96 \times 10^{-3} \text{ m}
\]

\[
= 4.96 \text{ mm}
\]
Exercise Problems

1. A block of dimensions 300 mm x 300 mm x 300 mm and mass 30 kg slides down a plane inclined at 30° to the horizontal on which there is a thin film of oil of viscosity 2.3 x 10⁻³ Ns/m². Determine the speed of the block if the film thickness is 0.03 mm. (Ans. 21.3 m/s)

2. Calculate the capillary effect in mm in a glass tube of 6mm diameter when immersed in (i) water, and (ii) mercury, both liquids being at 20° C. Assume σ to be 73 x 10⁻³ N/m for water and 0.5 N/m for mercury. The contact angles for water and mercury are 0° and 130° respectively.

3. Calculate the internal pressure of a 25 mm diameter soap bubble if the tension in the soap film is 0.5 N/m. (Ans. 80 N/m²)

4. A hydraulic ram 200 mm in diameter and 1.2 m long moves wholly within a concentric cylinder 100.2 mm in diameter, and the annular clearance is filled with oil of specific gravity 0.85 and kinematic viscosity 400 mm²/s. What is the viscous force resisting the motion when the ram moves at 120 mm/s?

5. Eight kilometers below the surface of the ocean the pressure is 81.7 MN/m². Determine the specific weight of sea water at this depth if the specific weight at the surface is 10.06 kN/m³ and the average bulk modulus of elasticity is 2.34 GN/m². Assume that g does not vary significantly. (Ans. 10.42 kN/m³)

6. A one square metre then plate is dragged at a velocity of 3 m/s on the top of a 5mm deep liquid of dynamic viscosity 20 centipoises. Assuming linear velocity variation in the liquid, find the drag force.
7. If the velocity distribution over a plate is given by

\[ u = \frac{3}{4}y - y^2 \]

where \( u \) is the velocity in m/s at distance \( y \) metres above the plate determine the shear stress at a distance of 0.15 m from the plate. Take the dynamic viscosity of the fluid as 0.834 Ns/m².

(Ans. 0.375 N/m²)

8. The volume of a liquid is reduced by 1% by increasing the pressure from 5 atmospheres to 125 atmospheres. Estimate the modulus of elasticity of the liquid.

9. A sliding fit cylindrical body 14.9 cm in diameter and 15 cm long and having a 1 kg mass drops vertically at a constant velocity of 5 cm/s inside a cylinder with 15 cm inside diameter, the space between the body and the cylinder is filled with oil. Estimate the viscosity of the oil.

(Ans. \( u = 1.4 \) Ns/m²)
CHAPTER 3

HYDROSTATICS

3.1 Introduction

Hydrostatics deals with the study of fluids that are at rest or are moving with uniform velocity as a solid body so that there is no relative motion between fluid elements. When there is no relative motion between fluid layers there is no shear stress in fluids at rest whatever the viscosity of the fluid. Hence only normal pressure forces are present in hydrostatics. Engineering applications of hydrostatic principles include the study of forces acting on submerged bodies such as gates, submarines, dams etc. and the analysis of stability of floating bodies such as ships, pontoons etc..

3.2 Pressure at a Point in a Static Fluid

In a fluid at rest, no tangential stresses can exist. The only forces between adjacent surfaces are pressure forces that are normal to the surfaces. Therefore the pressure at any point in a fluid at rest is the same in every direction. This is known as Pascal's Law. Pascal's Principle can be proved by considering a small wedge shaped fluid element at rest as shown in fig. 2.1. The thickness of the wedge perpendicular to the plane of the paper is dy.
Let $P_1$, $P_2$, and $P_3$ be the average pressures acting on the faces $ab$, $ac$ and $bc$ of the prism respectively. The weight of the fluid prism is $\frac{1}{2} \gamma \, dx \, dy \, dz$ where $\gamma$ is the specific weight of the fluid.

Since the fluid prism is in equilibrium, the equations of equilibrium will be:

X direction: $P_1 \, dz \, dy - P_3 \, dl \, dy \cos \alpha = 0$

but $dz = dl \cos \alpha$

so that, $P_1 \, dl \cos \alpha \, dy - P \, dl \, dy \cos \alpha = 0$

$\therefore \quad P_1 = P_3$

Z direction: $P_2 \, dx \, dy - P_3 \, dl \, dy \sin \alpha - \gamma \frac{dx \, dy \, dz}{2} = 0$

$dx = dl \sin \alpha$ and as $dx$, $dy$ and $dz$ all shrink to zero, the third term in the above equation becomes zero.

Thus $P_2 - P_3 = 0$

$\therefore \quad P_2 = P_3$

Then $P_1 = P_2 = P_3$
This shown that the pressure at a point in a static fluid is the same in all directions.

3.3 Basic Equation of Hydrostatics

The basic equation of Hydrostatics may be derived by considering the infinitesimal fluid parallelepiped in a static fluid shown in fig. 3.2. below.

Assuming the density of the fluid $\rho$ in the infinitesimal cube to be constant, the mass of the fluid is $\rho \cdot dx \cdot dy \cdot dz$. Let the pressure variation in the $x$, $y$, and $z$ directions be $\frac{\partial p}{\partial x}$, $\frac{\partial p}{\partial y}$, and $\frac{\partial p}{\partial z}$ respectively and let the entire fluid mass be subjected to acceleration of $a_x$, $a_y$, and $a_z$ in the $x$, $y$ and $z$ directions respectively.
Considering the equilibrium in the vertical (Z) direction:

\[ +p \, dx \cdot dy - (P + \frac{\partial P}{\partial Z} \, dz) \, dx \cdot dy - \rho \, g \, dx \, dy \, dz - a_z \cdot \rho \, dx \, dy \, dz = 0 \]

which reduces to

\[ \frac{\partial P}{\partial Z} = -\rho (a_z + g) \]

Similarly it can be shown that

\[ \frac{\partial P}{\partial y} = -\rho a_y \quad \text{and} \quad \frac{\partial P}{\partial x} = -\rho a_x \]

The total change in pressure is given by the total differential as follows:

\[ dp = \frac{\partial P}{\partial x} \, dx + \frac{\partial P}{\partial y} \, dy + \frac{\partial P}{\partial z} \, dz \]

Or

\[ dp = -\rho a_x \, dx - \rho a_y \, dy - \rho (a_z + g) \, dz \]

\[ \therefore \quad dp = -\rho [a_x \, dx + a_y \, dy + (a_z + g) \, dz] \quad (3.1) \]

Equation 3.1 is the basic equation of fluid statics applicable for both compressible and incompressible fluids.

3.4 Variation of Pressure with Elevation in a Static Incompressible Fluid

For a fluid at rest and subjected only to gravitational force, the accelerations ax, ay and az are zero. Eqn 3.1 thus reduces to:

\[ dp = -\rho g \, dz \quad (3.2) \]
Equation 3.2 holds true for both compressible and incompressible fluids. However for homogeneous and incompressible fluids, \( \rho \) is constant and eqn 3.2 may be integrated to give:

\[
p = -\rhogz + c
\]

where \( c \) is a constant of integration and is equal to the pressure at \( z = 0 \). In hydrostatics the law of variation of pressure with depth is usually written as;

\[
p = \rho gh + p_o
\]  \hspace{1cm} (3.3)

In Equation 3.3, \( h \) is measured vertically downward (i.e. \( h = -z \)) from a free surface, \( p \) is the pressure at a depth \( h \) below the free surface and \( p_o \) is the pressure at the free surface.

Equation 3.3 shows that for a fluid at rest, the pressures at the same depth from the free surface are equal. Hence in a homogeneous continuous fluid a surface of equal pressures is a horizontal plane.

Consider two points (1) and (2) at a depth of \( h_1 \) and \( h_2 \) in a tank containing a liquid, with density \( \rho \), at rest as shown in Figure 3.3. The pressure at (1) is \( p_1 = p_o + \rho gh_1 \). The pressure at (2) is \( p_2 = p_o + \rho gh_2 \). If \( h_1 = h_2 \), then \( p_1 = p_2 \).
For \( h_1 > h_2 \), the pressure difference between (1) and (2) is
\[
p_1 - p_2 = \Delta p = \rho g h_1 - \rho g h_2 = \rho g (h_1 - h_2) = \rho g \Delta z.
\]

\[
\Delta z = \frac{\Delta p}{\rho g}
\]
is the pressure difference between (1) and (2) expressed as a height of the liquid. This difference is also referred to as the pressure head difference. Thus, by dividing a pressure by the specific weight \( \gamma = \rho g \) of a fluid, the pressure can be expressed as height of fluid column.

3.5 Variation of Pressure with Elevation in Static Compressible Fluids

Since density varies with pressure in compressible fluids, the relation between density and pressure must be known in order to integrate the basic equation of fluid statics and obtain expressions for the variation of pressure with elevation in compressible fluids. The relation between pressure and density is dependent on the prevailing conditions. These conditions are: Constant temperature (i.e. isothermal), adiabatic and constant temperature gradient conditions.

Isothermal Condition: The relation between pressure, density and temperature for constant temperature condition is given by the perfect gas law: \( \frac{P}{\rho} = RT \). Substituting this in the basic equation of hydrostatics i.e. Equation 3.2:

\[
\frac{dp}{dz} = -\rho g = \frac{-\rho g}{RT}
\]

or

\[
\frac{dp}{p} = -\frac{g}{RT} \, dz
\]
Integrating from $p = p_1$ where $z = z_1$ to $p = p_2$ where $z = z_2$,

$$\log_e \left( \frac{p_2}{p_1} \right) = -\left( \frac{g}{RT} \right) (z_2 - z_1)$$

or

$$\frac{p_2}{p_1} = \exp \left( -\frac{g}{RT} (z_2 - z_1) \right) \quad (3.4)$$

**Adiabatic Condition:** Under adiabatic condition the relationship between pressure and density is given by $P/\rho^k = \text{constant} = p_1/\rho_1^k$,

so that

$$\rho = \rho_1 \left( \frac{p}{p_1} \right)^{1/k}$$

Substitution of the above in the basic equation 3.2 gives:

$$\frac{dp}{dz} = -(\rho_1 g/p_1^{1/k})$$

or $$dz = -(p_1^{1/k}/\rho_1 g) \cdot p^{1/k} \cdot dp$$

Integrating from $p = p_1$ when $z = z_1$, to $p = p_2$ when $z = z_2$,

$$z_2 - z_1 = -(p_1^{1/k}/\rho_1 g) \left[ \frac{p_2^{(k-1)/k}}{(k-1)/k} \right]_{p_1}^{p_2}$$

$$= -(k/(k-1)) \cdot (p_1^{1/k}/\rho_1 g) \left( p_2^{(k-1)/k} - p_1^{(k-1)/k} \right)$$

The above may be written as:

$$z_2 - z_1 = -(k/(k-1)) \left( \frac{p_2}{\rho_1} \right) \left\{ \left( \frac{p_2}{p_1} \right)^{(k-1)/k} - 1 \right\}$$
or, since \( p_1/\rho_1 = RT \) for any gas,

\[
z_2 - z_1 = -(k/(k-1)) RT_1/g \{(p_2/p_1)^{(k-1)/k} - 1\}
\]

or \( p_2/p_1 = \left\{1 - g(z_2 - z_1)/\left(RT_1 \left(\frac{k-1}{k}\right)\right)^{1/(k-1)}\right\}^{1/(k-1)} \) (3.5)

In the above equation, \( T \) is absolute temperature in \( \text{oK} \), \( R = 288 \text{ J kg}^{-1} \text{k}^{-1} = 288 \text{ m}^2/\text{s}^2/\text{oC} \) and \( k = 1.4 \) for adiabatic condition.

The temperature lapse rate - the rate of change of temperature with altitude - can be found for a diabatic conditions as follows:

Substituting the characteristic equation, \( q = p/RT \) in Equation 3.2 and rearranging,

\[
dz = -(RT/gp) \, dp
\]

For adiabatic condition, \( p/\rho^k = p_1/\rho_1^k \), and since \( p/\rho = RT \), substitution and rearranging gives:

\[
p = p_1 \left(T_1/T\right)^{k/(1-k)}
\]

Differentiating the above,

\[
dp = -(k/(1-k))p_1 \cdot T_1^{k/(1-k)} T^{-1/(1-k)} \, dT
\]

Substituting the values of \( p \) and \( dp \) in the equation for \( dz \),

\[
dz = (k/(1-k))(R/g) \, dT
\]

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Therefore, the temperature gradient is given by:

\[ \frac{dT}{dZ} = -\frac{(k-1)/k}{(g/R)} \]  

(3.6)

**Constant Temperature Gradient Condition:** Assuming that there is a constant temperature lapse rate (i.e., \( \frac{dT}{dZ} = \text{constant} \)) with elevation in a gas, so that its temperature drops by an amount \( \delta T \) for a unit change in elevation, then if \( T_1 \) = temperature at elevation \( Z_1 \), then \( T \) = temperature at elevation \( Z \) is given by:

\[ T = T_1 - \delta T (Z - Z_1) \]

Putting this in Equation 3.2 and noting that \( \rho/\varrho = RT \),

\[ \frac{dp}{dZ} = -\rho g/RT \]

or

\[ \frac{dp}{p} = (-\rho g/RT) dZ \]

Substituting the above value of \( T \),

\[ \frac{dp}{p} = -\{g/(RT_1 - \delta T(Z - Z_1))\} dZ \]

Integrating the above between limits \( P_1 \) and \( P_2 \) and \( Z_1 \) and \( Z_2 \),

\[ \frac{P_2}{P_1} = \{1 - (\delta T/T_1)(Z_2 - Z_1)\}^{\rho/(RT_1)} \]  

(3.7)

On the average, there is a temperature gradient of about 6.5°C per 1000 m in the atmosphere i.e., \( \delta T = 6.5°C \) per 1000 m = 0.0065°C.m⁻¹.

**3.6 Absolute and Gage Pressure**

A pressure may be expressed with reference to any arbitrary datum. It is usually expressed with respect to
Absolute zero (perfect vacuum) and local atmospheric pressure. When a pressure is expressed with respect to Absolute zero, the pressure is called Absolute pressure, $P_{abs}$. If a pressure is expressed with respect to local atmospheric pressure, it is called gage pressure, $P_{gage}$.

Figure 3.4 illustrates the concept of pressure datum.

\begin{figure}[ht]
\centering
\includegraphics[width=\textwidth]{pressure_datum}
\caption{Pressure and Pressure Datum}
\end{figure}

It is evident from fig. 3.4 that absolute pressure is always positive since there cannot be any pressure below absolute zero.

Gage pressure is positive if the pressure is greater than atmospheric pressure and negative if the pressure is lower than the atmospheric pressure. The following equation expresses the relationship between absolute, gage and atmospheric pressure;

$$P_{abs} = P_{atm} + P_{gage} \quad (3.8)$$

In equation 3.8 $P_{gage}$ may be positive or negative as the case may be. In hydrostatics, pressures are usually expressed as gage pressures unless otherwise specified.
Atmospheric pressure is also called Barometric pressure because the Barometer is the instrument used to measure the absolute pressure of the atmosphere. The simple barometer consists of an inverted tube closed at one end and immersed in a liquid with the open end down (Figure 3.5). If air is exhausted from the closed end of tube, the atmospheric pressure on the surface of the liquid in the container forces the liquid to rise in the tube.

If air is completely exhausted from the top portion of the tube, the liquid will rise in the tube to a height \( y \) and the only pressure on the liquid surface in the tube is the vapor pressure of the liquid, \( P_v \).

![Figure 3.5 The simple Barometer](image)

If \( \rho \) is the density of the liquid then the following equation is obtained from the variation of pressure in a static liquid.

\[
p_a = P_v + \rho gy = P_{atm}
\]

i.e. \( P_{atm} = \rho gy + P_v \)  \hspace{1cm} (3.9)

The vapour pressure \( P_v \) is very small compared to the atmospheric pressure. Hence equation 3.9 may be approximated to \( P_{atm} = \rho gy \). Thus the atmospheric pressure when expressed as
the depth of the liquid becomes \( y = \frac{P_{\text{atm}}}{\rho g} \) and \( y \) is called the pressure head. It follows from this that if a liquid with low density is used, \( y \) will be excessively large. Therefore, mercury is usually used in barometers mainly because its specific weight is very high thus enabling the use of short tube and also because its vapour pressure is negligibly small.

At sea level, \( y \) is 760 mm of mercury or 10.33 m of water. Atmospheric pressure at sea level is equal to 101.325 KN/m\(^2\) and is also called standard atmospheric pressure.

### 3.7 Measurement of Pressure

Pressure is always measured by the determination of a pressure difference. As mentioned earlier, liquid pressures are normally expressed with respect to the prevailing atmospheric pressure and are called gage pressures. Several devices are employed for measuring pressures. Some of these are discussed here.

#### 3.7.1 The Bourdon Gauge

The Bourdon gauge is a commercial instrument used to measure pressure differences (gage pressures) by the deformation of an elastic solid and may be employed when high precision is not required. It consists of a curved tube of elliptical cross-section closed at one end. The closed end is free to move while the other open end through which the fluid enters is rigidly fixed to the frame as shown in fig. 3.6.
The internal pressure intensity of the fluid tends to straighten the curved tube by an amount proportional to the pressure intensity. The deflection of the tube cause a pointer moving over a scale to undergo a corresponding angular displacement by means of a suitable gear and linkage arrangement. Zero reading is calibrated to correspond to local atmospheric pressure. All such gages required calibration.

3.7.2 Piezometer Column

A piezometer may be used to measure moderate positive pressures of liquids. It consists of a simple transparent tube open to the atmosphere in which the liquid can freely rise without overflowing as shown in Fig. 3.7. The height to which the liquid will rise in the tube indicates the pressure.
If the density of the liquid is $\rho$, then the pressure at A (gage pressure) is given by $P_A = \rho gh$. To reduce capillarity effects, the tube diameter should be at least 15 mm. Piezometers cannot be used to measure negative pressure since air will be sucked into the container.

3.7.3 Manometers

Manometers are devices used to measure the difference in pressure between a certain point and the atmosphere, or between two points neither of which is necessarily at atmospheric pressure. They are suitable for measuring high pressure differences both positive and negative, in liquids and gases.

The Common (simple) Manometer:

The simple manometer consists of a transparent U - tube connected to a pipe or other container containing fluid N (figure 3.8). The lower part of the U - tube contains liquid M which should be immiscible with N and is of greater specific gravity. The most frequently used manometer liquids are
mercury (specific gravity 13.6) and Alcohol (specific gravity 0.9).

Since the pressure in a continuous and homogenous fluid is the same at any two points in a horizontal plane, the pressures at K and L are equal for the equilibrium condition shown in Fig. 3.8. Thus if the specific weight of liquids N and M is \( \gamma_n \) and \( \gamma_m \) respectively one obtains the following:

\[
P_L = P_K
\]

or

\[
P_A + \gamma_N y_2 = \gamma_M y_1
\]

Thus

\[
P = \gamma_M y_1 - \gamma_N y_2 \tag{3.10}
\]

If liquid N is a gas, \( \gamma_N \) is negligible compared to \( \gamma_M \) and then \( P = \gamma_M y_1 \). In situations where the pressure to be measured is sub-atmospheric the arrangement may look like in Fig. 3.9.

Figure 3.8 The Simple Manometer
Figure 3.9

The manometric equation will now be \( p_A + y_2 \gamma_N + y_1 \gamma_M = 0 \).

**Differential Manometer**

A U-tube manometer is often used for measuring the difference in pressures between two containers as shown in Fig. 3.10. Such a manometer is sometimes referred to as differential manometer.

Considering points K and L an a horizontal plane in liquid M, \( p_K = p_L \) and this may be written as

\[
p_A + (y_3 + y_1) \gamma_N = p_B + (y_1 - y_2) \gamma_O + y_2 \gamma_M
\]

\[
\therefore p_A - p_B = (y_1 - y_2) \gamma_O + y_2 \gamma_M - (y_3 + y_1) \gamma_N
\]

**Micromanometers**

Micromanometers are used for measuring very small differences in pressure or precise determination of large pressure differences. A typical arrangement, shown in Fig.
3.11, consists of two immiscible gage liquids A and B which are also immiscible with the fluid C to be measured. Prior to connection to the two containers m and n, the heavier gauge liquid A fills the lower portion of the U-tube to the level 1-1 and liquid B is at level 0-0. Fig. 3.11 shows the equilibrium situation when the pressure at m is higher than at n.

Writing the manometric equation starting from m:

\[ P_m + (K_1 + \Delta y) \gamma_c + (K_2 - \Delta y + h/2) \gamma_B - h \gamma_A \]
\[ - (K_2 - h/2 + \Delta y) \gamma_B - (K_1 - \Delta y) \gamma_c = P_n \]

The above simplifies to:

\[ P_m + 2\Delta y \gamma_c - 2\Delta y \gamma_B + h \gamma_B - h \gamma_A = P_n \]

but \( \Delta y \cdot A = \frac{h}{2} a \) or \( 2\Delta y = \frac{a}{A} h \)
Substituting and rearranging;

\[ P_m - P_n = h(\gamma_A - \gamma_B(1 - \frac{a}{A}) - \gamma_c \frac{a}{A}) \]

The term in brackets is constant for a specific gauge and fluids and hence the pressure difference is directly proportional to \( h \).

**Example 3.1**

A closed tank is partly filled with water and connected to the manometer containing mercury \( (S = 13.6) \) as shown in the figure below. A gauge is connected to the tank at a depth of 4 m below the water surface. If the manometer reading is 20
cm, determine the gauge reading in N/m². What will be the
gauge reading when expressed as head of water in m?

![Diagram of air, water, and mercury columns with pressures labeled]

Solution:

Using the letter designation in the Figure, \( p_A = p_A' \)

\[
P_B = p_A' - 0.20 \gamma_M
\]

\( P_C = P_B \) and \( P_D = P_{gauge} = P_C + 4 \gamma_w \)

\[
P_D = p_A' - 0.20 \gamma_m + 4 \gamma_w
\]

\[
= 0 - 0.20 \gamma_w \cdot S_m + 4 \gamma_w = \gamma_w(-0.2S_m + 4)
\]

\[
= 9810 \frac{N}{m^3} (-0.2 \times 13.6 + 4) m = 9810(-2.72 + 4) N/m^2
\]

\[P_D = 9810 \times 1.28 = 12556.8 \text{ n/M}^2\]
Therefore, the gauge reading is 12556.8 N/m².  
When expressed as head of water, the gauge reading will be
\[
\frac{P}{\gamma_w} = \frac{12556.8 \text{ N/m}^2}{9810 \text{ N/m}^2} = 1.28 \text{ m}.
\]

Example 3.2

A manometer is mounted in a city water supply main pipe to monitor the water pressure in the pipe as shown below. Determine the water pressure in the pipe.

Solution:

\[ P_A = P_B \]

\[ \gamma_{Hg} \cdot 1 = P_p + 0.70 \gamma_w \]

\[ P_p = \gamma_{Hg} \times 1 - 0.7 \gamma_w + 13.6 \gamma_w - 0.7 \gamma_w \]

\[ = 12.9 \gamma_w = (12.9 \times 9810) \text{ N/m}^2 \]

\[ = 1,2655 \times 10^5 \text{ N/m} = 1.249 \text{ atmospheres} \]

(Note: 1 standard atmosphere = 1.01325 \times 10^5 \text{ N/m}^2)
Example 3.3 Calculate the height of liquid columns from the bottom of the tank in the three piezometer tubes shown in Figure E 3.3.

\[ \text{Pressure at C} = P_c \]
\[ P_c = 1.5 \times 9.81 \times 0.8 + 1.8 \times 9.81 \times 0.9 + 2 \times 9.81 \times 1.0 \]
\[ = 11.772 + 15.892 + 19.620 = 47.284 \text{ kN/m}^2 \]

Pressure head in terms of water = \( h_3 \)
\[ h_3 = \frac{47.284}{9.81} = 4.82 \text{ m} \]

Pressure at B = \( P_B \)
\[ P_B = 1.5 \times 9.81 \times 0.8 + 1.8 \times 9.81 \times 0.9 \]
\[ = 11.772 + 15.892 = 27.664 \text{ kN/m}^2 \]

Pressure head in terms of liquid with \( s = 0.9 \) is \( h_2 \)
\[ h_2 = \frac{27.664}{0.9 \times 9.81} = 3.13 \text{ m} \]
Pressure at A = \( P_A \)

\[
P_A = 1.5 \times 9.81 \times 0.8 = 11.772 \text{ kN/m}^2
\]

Pressure head in terms of liquid with \( s = 0.8 \) is \( h_1 \)

\[
h_1 = \frac{11.772}{0.8 \times 9.81} = 1.5 \text{ m}
\]

Therefore:

- Height of liquid surface in tube 1 from tank bottom
  \[= 1.5 + 1.8 + 2 = 5.3 \text{ m}\]
- Height of liquid surface in tube 2 from tank bottom
  \[= 3.13 + 2.0 = 5.13 \text{ m}\]
- Height of liquid surface in tube 3 from tank bottom
  \[= 4.82 \text{ m}\]

Example 3.4 Calculate the pressure at point A in Fig. E 3.4 and express it in terms of head of water.

Solution:

Starting from B,

\[
0 - 0.1 \times 9810 \times 13.6 + 0.8 \gamma_{\text{air}} - 0.4 \times 1.8 \times 9810 = P_A
\]

\[
- 13,341.6 + 0.8 \gamma_{\text{air}} - 7,063.2 = P_A
\]

\[P_A = 13,341.6 - 7,063.2 = 6,278.4 \text{ kN/m}^2\]

\[
h_1 = \frac{6,278.4}{0.8 \times 9.81} = 8.03 \text{ m}
\]
neglecting $\gamma_{air}$,

$$P_A = -20,404.8 \text{ N/m}^2 \quad \text{(vacuum)}$$

In terms of head of water, $h_A = \frac{-20404.8}{9810} = -2.08 \text{ m}$ of water

Example 3.5 Calculate the pressure difference between points A and B in the differential manometer shown in Figure E 3.5.

Solution:

Starting from A,

$$P_A + (x - 0.5)\gamma_w + 0.6\gamma_w - 0.6 \times 13.6\gamma_w - x\gamma_w = P_B$$

$$PA + x\gamma_w - 0.5\gamma_w + 0.6\gamma_w - 8.16\gamma_w - x\gamma_w = P_B$$

$$\therefore P_A - P_B = (8.16 - 0.1)\gamma_w = 8.06\gamma_w$$

$$= 8.06 \times 9.81 = 79.07 \text{ kN/m}^2$$

Example 3.6 In the two compartment closed tank shown in Fig. E 3.6, the pressure in the air in the left compartment is -26.7
kN/m² while that in the air in the right compartment is 19.62 kN/m². Determine the difference h in the levels of the legs of the mercury manometer. Specific gravity of mercury is 13.6.

Solution:

Since the pressure in a static fluid is the same in a horizontal plane, \( P_A = P_B \)

\[
P_A = P_c + 3.5 \times 9.81 = 19.62 + 3.5 \times 9.81 = 53.955 \text{kN/m}^2
\]

\[
P_B = P_d + 4.1 \times 0.9 \times 9.81 + h \times 13.6 \times 9.81
\]

\[
= -26.7 + 4.1 \times 0.9 \times 9.81 + h \times 13.6 \times 9.81
\]

\[
= -26.7 + 36.20 + 133.42 \times h
\]

\[
= 9.5 + 133.42 \times h
\]

Thus \( h = (53.955-9.5)/133.42 = 0.333 \text{ m} \)

\( = 33.3 \text{ cm} \)

Example 3.7 At an altitude \( Z_1 \) of 11,000 m the atmospheric temperature \( T \) is \(-56.6^\circ\text{C}\) and the pressure \( P \) is 22.4 \text{kN/m}^2.
Assuming that the temperature remains the same at higher altitudes, calculate the density of the air at an altitude of $Z_2$ of 15000 m. Assume $R = 287 \text{ J kg}^{-1} \text{ K}^{-1}$.

Solution:

Let $P_2$ be the absolute pressure at $Z_2$.

Since the temperature is constant:

$$P_2/P_1 = e^{-(R/T)(Z_2 - Z_1)}$$

Here, $P_1 = 22.4 \text{ kN m}^{-2} = 22400 \text{ N m}^{-2}$, $Z_1 = 11000 \text{ m}$, $Z_2 = 15000 \text{ m}$

$$R = 287 \text{ J kg}^{-1} \text{ K}^{-1}, \quad T = 56.6^\circ \text{C} = 216.5^\circ \text{K}:$$

$$P_2 = 22.4 \times 10^3 \exp \left\{ \frac{-9.81 \times (15000 - 11000)}{287 \times 216.5} \right\}$$

$$= 22.4 \times 10^3 \exp (-0.631) = 11.91 \times 10^3 \text{ N m}^{-2}$$

From the equation of state of a perfect gas, $P_2 = \rho_2 RT$

Therefore, the density of air at 15000 m is $\rho_2 = P_2/RT$

or $\rho_2 = 11.92 \times 10^3 / (287 \times 216.5) = 0.192 \text{ kg m}^{-3}$.

Example 3.8 Assuming that the temperature of the atmosphere drops with increasing altitude at the rate of 6.5$^\circ$ C per 1000 m, find the pressure and density at a height of 5000 m if the corresponding values at sea level are 101 kN m$^{-2}$ and 1.235 kg m$^{-3}$ respectively when the temperature is 15$^\circ$ C.

Take $R = 287 \text{ J kg}^{-1} \text{ K}^{-1}$.
Solution:

From Equation:

\[ P_2 = P_1 \left[ 1 - \left( \frac{\delta T}{T_1} \right) (Z_2 - Z_1) \right] \frac{g}{R \delta T} \]

\[ \delta T = 6.5^\circ C \text{ per } 1000 \text{ m} = 0.0065 \text{ K m}^{-1} \]

\[ T_1 = 15^\circ C = 288 \text{ K} \]

\[ Z_2 - Z_1 = 5000 - 0 = 5000 \text{ m}. \]

\[ \frac{g}{(R \delta T)} = \frac{9.81}{(287 \times 0.0065)} = 5.259 \]

\[ \therefore P_2 = 101 \times 10^3 \left[ 1 - (0.0065/288) \times 5000 \right] 5.259 = 53.82 \times 10^3 \text{ N m}^{-2} \]

From the equation of state

Density \[ \rho_2 = \frac{P_2}{RT_2} = \frac{P_2}{R(T_1 - \delta T(Z_2 - Z_1))} \]

\[ = \frac{53.82 \times 10^3}{287(288 - 0.0065 \times 5000)} \]

\[ = 0.734 \text{ kg m}^{-3} \]

3.8 Hydrostatic Forces on Surfaces

Plane and curved surfaces, immersed fully or partly in liquids, are subjected to hydrostatic pressure forces. It is, therefore, essential to determine the magnitudes, directions and locations of the hydrostatic pressure forces on surfaces as a first step in the analysis of the stability of a body fully or partly immersed in a liquid and in the design of hydraulic structures such as dams and gates.
3.8.1 Hydrostatic Force on Plane Surfaces

a) Horizontal Plane Surfaces:

The pressure intensity in a static fluid is the same at any two points in a horizontal plane surface. Therefore, a plane surface in a horizontal position at a depth \( h \) below the free surface in a fluid at rest will be subjected to a constant pressure intensity equal to \( \gamma h \), where \( \gamma \) is the specific weight of the fluid. The total pressure force on a small differential area is given by:

\[
dF_p = \gamma h \, dA
\]

The total pressure force on the entire horizontal plane surface with area \( A \) will be

\[
F_p = \int A \, \gamma h \, dA = \gamma \bar{h} A
\]

The force \( F_p \) acts normal to the surface and towards the surface. Since the pressure intensity is distributed uniformly over the plane surface, the total resultant force \( F_p \) acts through the centroid of the area and \( h = \bar{h} \), where \( \bar{h} \) is the depth from the free surface to the centroid. Thus, for horizontal plane surfaces, the centre of pressure \( C \) coincides with the centroid \( G \). The centre of pressure is the point on the immersed surface at which the resultant pressure force on the entire area is assumed to act.

b) Vertical Plane Surface

Consider a plane vertical surface of area \( A \) immersed vertically in a liquid (Fig. 3.12). Since the depth from the free surface
to the various points on the surface varies, the pressure intensity on the surface is not constant and varies directly with depth.

Consider also a narrow strip of horizontal area \( dA \), shown shaded in Fig. 3.12, at a depth \( h \) below the free surface. The pressure intensity on this area \( dA \) is \( \gamma . h \) and is uniform. The total pressure force on one side of the strip is thus

\[
dF_p = \gamma . h . dA
\]

The total pressure force on one side of the entire area \( A \) is

\[
F_p = \int \gamma h . dA = \gamma \int h . dA
\]

or

\[
F_p = \gamma . \bar{h}A
\]

where \( \bar{h} \) is the depth from the free surface to the centroid \( G \) of the area. Thus, as for a horizontal plane area, the magnitude of the resultant hydrostatic pressure force on a vertical plane area is obtained by multiplying the pressure intensity at the centroid \( G \), i.e \( \gamma . \bar{h} \), by the total area \( A \).
If the vertical area is not of a regular shape, the area may be divided into a finite number of small regular areas and the total hydrostatic pressure force determined as the sum of the pressure forces acting on these small areas.

The total pressure force $F_p$ acts normal to the vertical plane area and towards the area through the centre of pressure $C$. Since the pressure distribution on the area is not uniform, the centre of pressure and the centroid will not coincide. The depth $h_c$ to the centre of pressure may be obtained from the principle of moments. The moment of the elementary force $dF_p$, acting on the area $dA$ (Fig. 3.12) about axis 0-0 on the free surface is

$$dM = dF_p \cdot h = (\gamma \cdot h \cdot dA) \cdot h$$

The total moment of all elementary forces on the whole area is:

$$M = \int dM = \int \gamma \cdot h^2 \cdot dA$$

From the principle of moments, the sum of the moments of a number of forces about an axis is equal to the moment of their resultant about the same axis. Thus:

$$F_p \cdot h_c = M = \gamma \int h^2 dA$$

The term $\int h^2 dA$ may be recognized as the second moment of area about the free surface i.e. $I_{oo}$.

$$h_c = \frac{\gamma \cdot I_{oo}}{F_p}$$

i.e
using the parallel axis theorem of second moment of area,

\[ I_{oo} = I_G + A(h)^2 \]

where \( I_G \) is the second moment of area about the axis parallel to 0-0 and passing through the centroid G. Therefore,

\[ h_c = \frac{\gamma I_G + A(h)^2}{\gamma h A} \]

or

\[ h_c = \frac{I_G}{Ah} + \bar{h} \]

Thus, the centre of pressure C for vertical plane area is below the centroid by an amount equal to:

\[ I_G/Ah \]

. The moment of \( F_p \) about the centroid is:

\[ F_p \times GC = \rho gh - A \times \frac{I_c}{h - A} = \rho g I_c \], which is independent of depth of submergence.

c) Inclined Plane Surface

The analysis of the hydrostatic force on an inclined plane surface will be made by considering a plane surface of arbitrary shape and total area \( A \) inclined at an arbitrary angle \( \theta \) to the free surface as shown in Fig. 3.13. AB is the trace of the inclined surface the extension of which intersects with the free surface at 0. \( h_c \) and \( h_p \) are the depths from the free surface to the centroid C and centre of pressure CP of the area
respectively. \( y_c \) and \( y_p \) are the corresponding distances from 0 to \( C \) and \( CP \) respectively, measured along the inclined surface. It is required to determine the magnitude, direction and line of action of the resultant hydrostatic force \( F_p \) acting on one side of the area.

![Figure 3.13 Hydrostatic force on an inclined plane surface](image)

The magnitude of the force \( dF_p \) acting on an elementary area \( dA \) at a depth \( h \) below the free surface is given by

\[
dF_p = p \cdot dA = \rho gh \cdot dA = \rho g \cdot y \sin \theta \cdot dA
\]

The force \( dF_p \) acts normal to the plane surface. The resultant hydrostatic force \( F_p \) is the sum of all elementary forces \( dF_p \) which are parallel to each other.
Thus \( F_p = \int dF_p = \rho g \sin \theta \int y dA \)

But \( \int y \, dA \) is the first moment of area \( A \) about axis through 0 and is equal to \( y_c A \) and since \( y \sin \theta = c h \), the above equation for \( F_p \) becomes:

\[
F_p = y \sin \theta y_c A = \gamma h_c A \quad 3.11
\]

\( \gamma h_c \) is the pressure intensity at the centroid of the inclined plane area. This shows that the magnitude of the resultant hydrostatic force on an inclined plane area is equal to the product of the area and the pressure intensity at the centroid of the area. The force \( F_p \) acts normal to the plane surface and towards the surface.

The resultant force \( F_p \) acts through the centre of pressure \( CP \) of the submerged plane area. The location of \( CP \) is determined using the principle of moments for a parallel force system. In Fig. 3.13 let the axis through 0 coinciding with the free surface be the axis of moments. The moment of force \( dF_p \) about this axis is equal to \( dM_o \) which is given by

\[
dM_o = y \, dF_p = y \, \rho g \, y \sin \theta \, dA = \rho g \sin \theta \, y^2 \, dA
\]

The moment of the resultant force \( F_p \) about the axis of moments will be equal to the sum of all elemental moment \( dM_o \). i.e.

\[
F_p \, y_{cp} = \int dM_o = \rho g \sin \theta \int y^2 \, dA = \gamma \sin \theta \, I_\infty
\]

Where \( I_\infty \) is the second moment of the plane area about axis 0-0.
Thus \( y_{cp} = \frac{\gamma \sin \theta \ I_{oo}}{F_p} = \frac{\gamma \ sin \theta \ I_{oo}}{\gamma \ sin \theta \ \ y_c \ A} = \frac{I_{oo}}{y_c \ A} \)

Using the parallel axis theorem,

\[ I_{oo} = I_c + y_c^2 \ A \]

Where \( I_c \) is the second moment of area about an axis parallel to 0-0 and passing through the centroid \( c \).

\[ y_{cp} = \frac{I_c + y_c^2 \ A}{y_c \ A} = y_c + \frac{I_c}{y_c \ A} \]

Thus \( (3.12) \)

This shows that the centre of pressure is always below the centroid of the area. The same has been shown for vertical plane surfaces.

The depth of the centre of pressure below the free surface is \( h_{cp} = y_{cp} \ \sin \theta \) . Substituting this and the value of \( y_c = h_c/Sin\theta \) in Eqn. \( , \) the following equation is obtained for the depth to the centre of pressure.

\[ h_{cp} = h_c + \frac{I_c \ \sin^2 \theta}{h_c \ A} \] (3.13)

When the surface area is symmetrical about its vertical centroidal axis, the centre of pressure CP always lies on this symmetrical axis but below the centroid of the area. If the area is not symmetrical, an additional coordinate, \( x_{cp} \), must be fixed to locate the centre of pressure completely.
Referring to Figure 3.14, and using moments,

\[ X_{cp} \int_A dF_p = \int_A dF_p x \]

or

\[ X_{cp} \rho g y_c \sin \theta A = \int_A \rho g y \sin \theta dA x \]

\[ \therefore X_{cp} = \frac{1}{Ay_c} \int_A x y \]

Figure 3.14  Centre of pressure of an asymmetrical plane surface

The location of the centroid C and the magnitude of the second moment of area about the centroidal axis of some common geometrical shapes is given in Table 3.1.

3.8.2 Hydrostatic Force on Curved Surfaces

The total hydrostatic force on a curved surface immersed in a liquid can not be directly determined by the methods developed for plane surfaces. For plane surfaces, the pressure forces on elementary areas act perpendicular to the surface and hence are parallel to each other. Consequently,
it is easier to obtain the resultant force by a simple summation of the elementary forces. In the case of a curved surface each elementary force acts perpendicular to the tangent of the elementary area and because of the curvature of the surface the direction of each elementary force is different. As a result, the usual procedure is to determine the horizontal and vertical components of the resultant force and then add them vectorially to obtain the magnitude, direction and location of the line of action of the resultant hydrostatic force.

Consider the curved surface BC of unit width shown in Figure 3.15.

Figure 3.15 Hydrostatic force components on curved surfaces.

The elementary force \( dF \) acting on the elementary area \( dA \) has a horizontal component \( dF_x \) and a vertical component \( dF_y \). The pressure intensity on \( dA \) is \( \rho gh \).

\[
\begin{align*}
\text{The total hydrostatic force on } dA &= dF = \rho gh \ dA \\
\text{The horizontal component of } dF &= dF_x = \rho gh \ dA \ \text{Cos}\theta \\
\text{The vertical component of } dF &= dF_y = \rho gh \ dA \ \text{Sin}\theta \\
\end{align*}
\]

But \( dA \ \text{Cos}\theta = dA_v = \) The projection of \( dA \) on the vertical plane
and $dA \sin \theta = dA_h = \text{The projection of } dA \text{ on the horizontal plane.}$

The components of the total hydrostatic force in the $x$ and $y$ directions are $F_x$ and $F_y$ respectively and are given by:

$$F_x = \int_A dF_x = \int_A \rho gh \, dA \cos \theta = \rho g h \, A_v$$

$$F_y = \int_A dF_y = \int_A \rho gh \, dA \sin \theta = \rho g \int_A dV$$

where: $A_v$ is the projection of the whole curved surface BC on the vertical plane, i.e. BD

dV is the volume of the water prism (real or virtual) extending over the area $dA$ to the free surface.

i.e $F_y = \rho g V$

Thus:
The horizontal component, $F_x$, of the resultant hydrostatic force on a curved surface BC is equal to the product of the vertically projected area of BD and the pressure intensity at the centroid of the vertical area BD. The Force $F_x$ passes through the centre of pressure of the vertically projected area BD.

The vertical component, $F_y$, of the resultant hydrostatic force on a curved surface BC is equal to the weight of the water (real or virtual) enclosed between the curved surface BC, the vertical BD and the free surface CD. The force component $F_y$ acts through the centre of gravity of the volume.
The resultant force $F$ is given by:

$$F = \sqrt{F_x^2 + F_y^2}$$  \hspace{1cm} (3.14)

$F$ acts normal to the tangent at the contact point on the surface at an angle $\alpha$ to the horizontal, where

$$\alpha = \tan^{-1} \left( \frac{F_y}{F_x} \right)$$  \hspace{1cm} (3.15)

### 3.8.3 Pressure Diagrams:

The resultant hydrostatic force and centre of pressure for regular plane areas could be determined from pressure distribution diagrams such as those shown in Figure 3.16

![Pressure Diagrams](image)

**Figure 3.16** Pressure diagrams

In Fig. 3.16(a) the surface AB is horizontal and the pressure intensity is uniform over the area of the horizontal surface AB. The total hydrostatic thrust on AB is equal to the volume of the pressure prism, which is the product of the uniform pressure intensity $\rho gh$ and the area A, and acts through the centroid of the area.
In Fig. 3.16(b), AB may be assumed to be rectangular with width b perpendicular to the plane of the paper. The pressure distribution is trapezoidal with intensity \( \rho g h_1 \) at A and \( \rho g h_2 \) at B. The total hydrostatic force on AB is equal to the volume of the pressure prism and is given by:

\[
F = \frac{(\rho g h_1 + \rho g h_2)}{2} (h_2 - h_1) \cdot b
\]

The centre of pressure is the centroid of the pressure prism. It may be located by dividing the prism into a rectangular and triangular prism. For the rectangular prism, the centroid is at \((h_2 - h_1)/2\) above B and for the triangular prism it is at \((h_2 - h_1)/3\) above B. The centroid of the trapezoidal prism can then be found from the principle of moments.

### 3.8.4 Tensile Stress in a Pipe

Pipes are conduits of circular cross-section that are used to transport fluids. During this transport process, a certain amount of internal pressure is necessary to make the flow possible. This pressure may be supplied by gravity flow or by an external input of energy by means of a pump. The internal pressure produces tensile stresses in the pipe walls. Both longitudinal and circumferential (or hoop) stresses exist in pipes. However, the circumferential stresses are more important since they are twice the longitudinal stresses. In pipe flows, the problem is to determine either the required wall thickness of the pipe necessary to resist a certain pressure or the allowable pressure for a given wall thickness of a given pipe material. A circular pipe with internal radius \( r \), wall thickness \( t \) and having a horizontal axis is in tension around its periphery as shown in Fig. 3.17. A 1 metre long section of pipe, i.e. the ring between two planes normal to the axis and 1 metre apart, is considered for the analysis of the problem. Considering one-half of this ring as a free body, the
tensile forces per metre length at the top and bottom are $T_1$ and $T_2$ respectively.

Figure 3.17 Tensile stress in pipes.

The horizontal component of the pressure force acts through the centre of pressure $CP$ of the projected area and is given by:

$$F = p \cdot 2r \cdot l$$

Where $p$ is the pressure intensity at the pipe centre and $r$ is the internal radius $r$.

Strictly speaking, $T_1$ is smaller than $T_2$. But for high internal pressure, the centre of pressure $CP$ may be taken to coincide with the pipe centre $c$ and $T_1$ may be approximated to be equal to $T_2$ without serious error. Thus summing forces in the horizontal direction:

$$T_1 + T_2 = F = 2pr$$

since $T_1 = T_2 = T$, $2T = 2pr$ and $T = pr$
where $T$ is the tensile force per metre length of pipe. For wall thickness $t$, the circumferential stress $\sigma$ in the pipe wall will be

$$\sigma = \frac{T}{t \times 1} = \frac{P \cdot r}{t}$$

(3.16)

For an allowable tensile stress $\sigma_{all}$, the required wall thickness $t$ will be:

$$t = \frac{P \cdot r}{\sigma_{all}}$$

Where $P$ is in $N/m^2$, $\sigma_{all}$ is in $N/m^2$, $r$ is in cm and $t$ is in cm.

For large variations in pressure between top and bottom of pipe, i.e. when $Z_c = p_c / \gamma \leq 10r$, the centre of pressure has to be computed for which the following two equations are necessary:

From $\Sigma F_h = 0$ : $T_1 + T_2 = F = 2p \cdot r$

$$\Sigma M_2 = 0 : 2rT_1 - 2pr(r - e) = 0$$

From which:

$$T_1 = p(r - e)$$

and

$$T_2 = 2pr - T_1 = 2pr - p(r - e) = p(r + e)$$

Obviously, $T_1 > T_1$ and $T_2$ must be used for further computations. The eccentricity $e$ may be obtained as follows:
But, \( y_c = Z_c = \frac{p_c}{\gamma} \), taking the area as a vertical area.

\[
e = \frac{I_{xc}}{y_c A} = \frac{1 \cdot (2r)^3}{12 \cdot y_c (2r)\ell}
\]

\[
\therefore e = \frac{8r^3 \gamma}{12 \cdot p_c \cdot 2r} = \frac{r^2 \gamma}{3 \cdot p_c}
\]

from \( \sigma = \frac{T_2}{t} = \frac{p(r + e)}{t} \),

\[
t = \frac{p(r + e)}{\sigma_{all}} = \frac{p(r + r^2 \gamma/3p_c)}{\sigma_{all}} \tag{3.17}
\]

In a thin spherical shell subjected to an internal pressure the stress in its wall may be found, neglecting the weight of the fluid in the sphere, by considering the forces on a free body consisting of a hemisphere, cut from the sphere by a vertical plane as shown below.

The component of the pressure force \( F \) is:

\[
F = p \Pi r^2, \text{ where } r \text{ is the internal radius of the sphere and } p \text{ is the internal pressure.}
\]

If \( \sigma_t \) is the stress in the wall, then for equilibrium:

\[
\sigma_t - 2\pi r \cdot t - p\Pi r^2 = 0
\]
\[
\sigma_t = \frac{Pr}{2t}
\]  

(3.18)

\(\sigma_t\) is just half of the circumferential stress \(\sigma\) given by Eqn.3.16. For a pipe closed at one end, \(\sigma_t\) will be the longitudinal stress in the pipe wall.

Example 3.7

A vertical rectangular gate AB shown in Figure E.37 has a width of 1.5 m. The gate is hinged on its upper edge at A. Determine the moment \(M\) at A required to just hold the gate from opening.

![Figure E 3.7](image)

Solution: Referring to Figure E.37

The centroid C of AB is at \((4.5 - 1) = 3.5\) m below the water surface i.e. \(h_c = 3.5\) m.

The hydrostatic force on gate AB is \(F_p\) and will act through CP normal to AB.

\[F_p = \gamma_w h_c A = 9.81 \text{ kN/m}^3 \cdot 3.5 \text{ m} \cdot (2 \times 1.5) \text{ m}^2 = 103.005 \text{ kN}\]
The location of CP is obtained from:

\[ Y_{cp} = Y_c + \frac{I_c}{Y_c A} = 3.5 + \frac{1.5 \times 2^3}{12 \times 3.5 \times 2 \times 1.5} = 3.595 \text{ m} \]

Taking sum of moments about A,

\[ \sum M_A = M - 103.005 \times 3.595 = 0 \]
\[ \therefore M = 370.3 \text{ kN.m Clockwise} \]

Example 3.8

The 2 m wide and inclined rectangular gate AB shown in Figure E 3.8 is hinged at B. The gate is uniform and weighs 24 kN. Determine

a) The magnitude and location of the hydrostatic forces on each side of the gate.
b) The resultant of the hydrostatic forces.
c) The force F required to just open the gate.

Figure E 3.8
Solution:

a) Let the hydrostatic force on the left side of gate AB be \( F_l \) and that on the right side of \( F_r \).

The gate AB is 5 m long and its centroid C is at a depth of 1.5 from B.

Thus, \( F_l = \gamma_w h_{ct} A = 9.81 \times (3 + 1.5) \times (5 \times 2) = 441.45 \text{ kN} \)

\( F_l \) acts normal to AB through the center of pressure of the left side of AB which is located at \( Y_{cpl} \) from O1.

\[
Y_{cpl} = Y_{cl} + \frac{I_c}{Y_{cl} A}
\]

\( Y_{cl} = CO_1 = 7.5 \text{ m} \)

Therefore,

\[
Y_{cpl} = 7.5 + \frac{2 \times 5^3}{12 \times 7.5 \times 5 \times 2} = 7.5 + 0.278 = 7.778 \text{ m}.
\]

Similarly,

\[
F_r = \gamma_w h_{cr} A
= 9.81 \times (1.5 + 1.5) \times (5 \times 2) = 294.3 \text{ kN}
\]

\( F_r \) acts normal to AB through the center of pressure of the left side of AB which is located at \( Y_{cpr} \) from O2.

\[
Y_{cpr} = Y_{cr} + \frac{I_c}{Y_{cr} A}
\]

\( Y_{cr} = CO_2 = 5 \text{ m} \)
Therefore \( Y_{cp} = 5 + \frac{2 \times 5^3}{12 \times 7.5 \times 5 \times 2} = 5 + 0.417 = 5.417 \ m \)

The positions of the forces \( F_l \) and \( F_r \) shown below:

b) The resultant of \( F_l \) and \( F_r \) is \( F_R \) and acts parallel to \( F_l \) and \( F_r \) in the direction of the greater force \( F_l \) normal to \( AB \).

Thus, \( F_R = 441.45 - 294.3 = 147.15 \ kN \)

Taking moments about the hinge \( B \), the location of \( F_R \) from \( B \) is found as

\[
\bar{X} = \frac{441.45 \times (2.5 + 0.278) - 294.3 \times (2.5 + 0.417)}{147.15} = 2.5 \ m
\]

i.e \( F_R \) acts through \( C \).

c) The force \( F \) required to just open the gate will be obtained by taking moments of the forces shown in the sketch below about the hinge \( B \).
Example 3.9
A triangular opening in the form of an isosceles triangle, with dimensions shown in Figure E 3.9 and with its axis of symmetry horizontal, is closed by a plate. Water stands at 9 m from the axis of symmetry. Determine the resultant hydrostatic force on the plate and its centre of pressure.

Solution:

Plate area = $1/2 \times 6 \times 3 = 9 \text{ m}^2$.
Depth to centroid of plate = $9 \text{ m} = h_c$
Total hydrostatic force on plate = $F = \gamma_h h_c A$

\[
F = \frac{147.15 \times 2.5 + 48}{4} = 103.97 \text{ kN}
\]

The force $F$ acts through the centre of pressure CP and normal to the plate.

To determine the vertical location of the centre of pressure,
it is necessary to determine the second moment of area of the triangle about axis AD since \( y_{cp} = Y_c + \frac{I_c}{Y_c A} \).

The following steps will be used to determine \( I_c \).

i) Split the plate into two triangles: ABD and ADE.

ii) Determine the second moment of area of the two triangles ABD and ADE separately about line AD.

iii) Add the results in (ii) to obtain the second moment of area \( I_c \) of triangle ABC about axis AD.

Thus, second moment of area of triangle ABD about AD is given by

\[
\frac{bh^3}{12} = \frac{6 \times 1.5^3}{12} = 1.6875 \text{ m}^4
\]

Second moment of area of triangle ADE about AD is

\[
\frac{bh^3}{12} = \frac{6 \times 1.5^3}{12} = 1.6875 \text{ m}^4
\]
Therefore, the second moment of area of triangle ABC about the centroidal axis AD is:

\[ I_c = 1.6875 \times 9 + 1.6875 = 3.375 \, \text{m}^4 \]

The depth \( h_{cp} \) to the centre of pressure CP will be:

\[ h_{cp} = h_c + \frac{I_c}{h_c A} = 9 + \frac{3.375}{9 \times 9} = 9.042 \, \text{m}. \]

In the horizontal direction the centre of pressure CP is located on the vertical passing through the centroid C i.e. the vertical at \( 6/3 = 2 \, \text{m} \) from BE.

Example 3.10

A vertical, symmetrical trapezoidal gate with its upper edge located 5 m below the free surface is shown in Figure E 3.10, Determine the total hydrostatic force and its centre of pressure.

Figure E 3.10
Solution:

The total hydrostatic force = \( F = \gamma h_c A \)

The depth to centroid \( C \) = \( h_c = 5 + \frac{2(2 \times 1 + 3)}{3(1 + 3)} = 5.833 \text{ m} \)

The area = \( A = 2 \times (3 + 1)/2 = 4 \text{ m}^2 \)

Therefore, \( F = 9.81 \times 5.833 \times 4 = 228.89 \text{ kN} \)

The location of the centre of pressure is obtained from:

\[
 y_{cp} = y_c + I_c / (A \cdot y_c)
\]

\[
 I_c = \frac{h^3(a^2 + 4ab + b^2)}{36(a + b)}
\]

where \( a \) is the length of the shorter side, \( b \) the length of the longer side and \( h \) the distance between these parallel sides.

Thus,

\[
 I_c = \frac{2^3(1^2 + 4 \times 1 \times 3 + 3^2)}{36(1 + 3)} = \frac{8(22)}{36 \times 4} = 1.222 \text{ m}^4
\]

\[
y_c = h_c = 5.833 \text{ m}
\]

Therefore, \( y_{cp} = 5.833 + \frac{1.222}{4 \times 5.833} = 5.885 \text{ m} \) below the free surface.
Example 3.11

An inverted semicircular plane gate shown in Figure E 3.11 is installed at 45° inclination as shown. The top edge of the gate is at 3 m below the water surface. Determine the total hydrostatic force and the centre of pressure.

Solution:

The total hydrostatic pressure = \( F = \gamma \cdot h_c \cdot A \)

\[ h_c = y_c \cdot \sin 45° \]

\[ y_c = \frac{3}{\sin 45°} + \frac{4r}{3\pi} = 4.24 + \frac{4 \times 4}{3\pi} = 5.94 \text{ m}. \]

Therefore, \( h_c = 5.94 \cdot \sin 45° = 4.20 \text{ m}. \)
Area of gate = \( A = \frac{1}{2} \pi r^2 = \frac{1}{2} \times \pi \times 4^2 = 25.13 \, \text{m}^2 \)

Thus, \( F = 9.81 \times 4.20 \times 25.13 = 1.035 \, \text{MN} \)

\[
\frac{y_{cp}}{y_c} = 5.94 + \frac{0.11 \times 4}{5.94 \times 25.13} = 6.13 \, \text{m.}
\]

and \( h_{cp} = y_{cp} \sin 45^\circ = 4.33 \, \text{m} \)

Example 3.12

A log hods water as shown in Figure E 3.12. Determine

a) The force pushing against the obstruction (dam) per metre length of log.

b) The weight of the log per metre length

c) The specific gravity of the log.

Solution:

a) The force pushing against the dam is equal to the horizontal component of the hydrostatic force acting on the long.
Since the hydrostatic forces acting on surfaces BC and CD are equal and opposite to each other, they cancel out. Hence, the net hydrostatic force acting on the log is the horizontal component acting on curved surface AB.

\[ F_H = \gamma_c h_c A \]

\[ = (9.81 \times 0.8) \times 0.5 \times 1 \times 1 = 3.924 \text{ kN} \]

b) The weight of the log is equal to the net vertical hydrostatic force on the log. The vertical component of the hydrostatic force is composed of the vertically upward force on surface BCD and the vertically downward force on surface AB.

On surface AB, the vertical force \( F_1 \) is equal to the weight of the oil supported by it. i.e.

\[ F_1 = [(1 \times 1) - \frac{\pi}{4} \times 1^2] \times 1 \times 0.8 \times 9.81 = 1.684 \text{ kN downwards} \]

On surface BCD the vertical force \( F_2 \) equals the weight of water and oil (real and virtual) supported by it. Thus

\[ F_2 = (2 \times 1 \times 0.8 \times 9.81) + \frac{\pi}{2} \times 1^2 \times 1 \times 9.81 \]

\[ = 15.696 + 15.409 = 31.105 \text{ kN, upwards} \]

Net vertical hydrostatic force = \( F_2 - F_1 \),

\[ F_2 - F_1 = 31.105 - 1.684 = 29.421 \text{ kN upwards} \]

Therefore, the weight of the log per metre is 29.421 kN.

c) To determine the specific gravity of the log, determine its density first.
\[ \rho_l \times 9.81 \times \pi \times 1^2 \times 1 = 29421 \text{ N} \]

from which, \[ \rho_l = \frac{29421}{\pi \times 9.81} = 954.6 \text{ kg} \]

Specific gravity of log, \[ S_l = \frac{\rho_l}{\rho_w} = \frac{954.6}{1000} = 0.955 \]

Example 3.13

Referring to Figure E 3.13, calculate the force F required to hold the 1.4 m wide gate AB in a closed position if \( y = 0.8 \text{ m} \).

![Figure E 3.13](image)

Solution:

Let the pressure at the interface AA' between oil and water be \( P_A \).

Starting from the open end of the manometer, the hydrostatic pressure variation gives:
\[
0 + y \times 3\gamma_w - 1.6\gamma_w = P_A \\
\text{or } 0.8 \times 3\gamma_w - 1.6\gamma_w = P_A
\]

Thus:
\[
P_A = (2.4 - 1.6)\gamma_w = 0.8\gamma_w
\]

i.e. head at interface = 0.8 m of water = \[\frac{0.8}{0.7} = 1.143\] m of oil.

Referring to the following sketch:

The elementary hydrostatic force \(dF_p\) acting normal to the elementary area of length \(rd\theta\) of the gate is given by:

\[
dF_p = \gamma_c \cdot h_c \cdot A = 0.7\gamma_w(1.143 - r\cos\theta)(1.4 \times rd\theta)
\]

\[
= 0.7\gamma_w[1.143 - 0.8\cos\theta] \times 1.4 \times 0.8 d\theta
\]

For equilibrium, the sum of moments about the hinge B is zero.
Thus: \( F = 5714 \, N \)
Solution

Referring to the following sketch:

a) The horizontal component of the hydrostatic force, $F_H$, is the force acting on the projected area BB'
Therefore,

\[ F_H = \gamma_w \cdot h_c \cdot A = 9.81 \times 2 \times (4 \times 4.2) = 329.616 \text{ kN} \]

\[ y_{cp} = y_c + \frac{I_c}{y_c A} = 2 + \frac{4.2 \times 4^3}{12 \times 2 \times 4.2 \times 4} = 2.667 \text{ m} \]

Thus: \( \bar{y} = 4 - 2.667 = 1.333 \text{ m}. \)

The vertical component, \( F_v \), of the hydrostatic force is equal to the weight of the volume of fluid bounded in AA'B and acts through the centroid of this volume. i.e.

\[
F_v = \int_0^4 9.81.x \cdot d_y \times 4.2 \\
= \int_0^4 41.202 \cdot \frac{y^2}{4} \cdot d_y = 10.301 \cdot \frac{y^3}{3} \bigg|_0^4 = 219.744 \text{ kN}
\]

To locate the line of action of \( F_v \),

\[
F_v \cdot \bar{x} = \int_0^4 (4 - y) \cdot d_x \cdot 4.2 \times 9.81 \cdot x = 41.202 \int_0^4 (4 - y) \cdot x \cdot d_x
\]

\[
= 41.202 \int_0^4 (4 - 2 \cdot x^{1/2}) \cdot x \cdot d_x, \text{ since } y = 2 \sqrt{x}
\]

\[
= 41.202 \left[ 2x^2 \bigg|_0^4 - \frac{2}{5} \cdot x^{5/2} \bigg|_0^4 \right]
\]

\[
= 41.202 \left[ 32 - 25.6 \right] = 263.693
\]

Therefore, \( \bar{x} = \frac{263.693}{219.744} = 1.2 \text{ m} \)
b) Summing moments about $A$ to obtain the moment $M$ required to hold the gate $AB$,

$$\sum M_A = M - P_v \bar{x} - P_h \bar{y} = 0$$

from which, $M = P_v \bar{x} + P_h \bar{y} = 263.693 + 329.616 \times 1.333$

$$\therefore M = 703.07 \text{ kN-m}$$

3.9 Buoyancy and Stability of Submerged and Floating Bodies

Since the pressure in a fluid at rest increases with depth, the fluid exerts a resultant upward force on any body which is fully or partially immersed in it. This force is known as the Buoyant Force.

The principles of buoyancy and floatation, established by Archimedes (288-212 B.C), state that

i) a body immersed in a fluid in buoyed up by a force equal to the weight of the fluid displaced by the body and

ii) A floating body displaces its own weight of the fluid in which it floats.

These principles can be easily proven using the principle of hydrostatic force on surfaces.

3.9.1 Buoyant Force

Consider a body $ABCD$, shown in Fig. 3.18, submerged in a liquid of constant density $\rho$.

Referring to Fig. 3.18, $A'C'$ is the projection of the body on a horizontal plane and $B'D'$ is its projection on a vertical
Figure 3.18 Buoyant Force

plane. Force \( F_x \) acting to the right is the horizontal component of the hydrostatic force on surface BAD and force \( F'_x \), acting to the left is the horizontal component of hydrostatic force on surface BCD. Both \( F_x \) and \( F'_x \) are equal to the force acting on vertical plane surface B'D' and since they are equal, opposite and collinear, they cancel each other. Hence, the resultant horizontal hydrostatic force on a submerged body is zero.

Force \( F_z \) is the downward, vertical component of the hydrostatic force acting on surface ABC. \( F_z \) is equal to the weight of the liquid volume \( A'A BC C'A' \) i.e. \( F_z = \rho g \cdot \text{Vol. } A'A BCC'A' \).

Similarly, \( F'_z \) is the upward vertical component of the hydrostatic force on surface ADC and is equal to the weight of the liquid volume \( A'A DC C'A' \), i.e. \( F'_z = \rho g \cdot \text{Vol. } A'A DC C'A' \).

The net upward force is the buoyant force \( F_B \), which is

\[ \begin{align*}
F_B &= F'_z - F_z \\
&= \rho g \cdot \text{Vol. } A'A DC C'A' - \rho g \cdot \text{Vol. } A'A BC C'A'
\end{align*} \]

or

\[ F_B = \rho g \cdot \text{Vol. } ABCD \]  \( (3.19) \)
Thus, the buoyant force $F_b$ is the weight of the liquid displaced by the body and acts vertically upwards through the centre of buoyancy which is coincident with the centroid of the volume of the displaced liquid. Similar considerations show that for a body partially immersed in a liquid, the buoyant force is equal to the weight of the displaced liquid.

Considering the vertical equilibrium of a body submerged in a fluid, the condition of floatation of the body depends upon the relative magnitude of the weight of the body and the buoyant force. If the body is heavier than the weight of the fluid it can displace, it will sink to the bottom unless it is prevented from doing so by the application of an upward supporting force. If the weight of the body is lighter than the weight of the liquid it can displace when completely submerged in the fluid, it will rise above the surface to a position such that the weight of the displaced liquid is equal to the weight of the body.

The principle of buoyancy can be used to determine the weight, volume and consequently the specific weight and specific gravity of an object by weighing the object in two different fluids of known specific weights. Consider an object suspended and weighed in two fluids with specific weights $\gamma_1$ and $\gamma_2$ as shown in Fig. 3.19. Let the weight of the object be $W$ and its volume $V$.

Vertical equilibrium of forces in Figure 3.19(a) gives:

$$F_1 + \gamma_1 V = W$$

Vertical equilibrium of forces in Figure 3.19(b) gives:

$$F_1 + \gamma_2 V = W$$
Equating the above two equations and rearranging:

\[ V (\gamma_2 - \gamma_1) = F_1 - F_2 \]

From which

\[ F = \frac{F_1 - F_2}{\gamma_2 - \gamma_1} \]  \hspace{1cm} (3.20)

Substituting the value of \( V \) from the above equation in any of the two equilibrium equations, the following equation for the weight of the body may be obtained.

\[ W = \frac{F_1 \gamma_2 - F_2 \gamma_1}{\gamma_2 - \gamma_1} \]

The specific weight of the body will be \( \gamma = W/V \). It should be noted that the body should not be weighed in a liquid in which it dissolves.
The hydrometer, which is an instrument used to determine the specific gravity of liquids, is constructed on the basis of the principle of buoyancy. It consists of a closed glass tube with an enlarged bulb shape at the bottom in which lead shots are kept to allow it to float vertically when immersed in a liquid. The hydrometer sinks to different depth when immersed in liquids of different specific gravities, sinking deeper in lighter liquids than in heavier liquids. The graduations on the stem, from which the specific gravities are read directly at the meniscus, are obtained by calibration in liquids of known specific gravities. The reading 1.00 corresponds to that of distilled water.

3.9.2 Stability of Submerged Bodies

A submerged or a floating body is said to be stable if it comes back to its original position after a slight disturbance. The stability of a submerged body depends upon the relative position of its centre of gravity and its centre of buoyancy both of which have fixed positions.

Consider the three possible relative positions of centre of buoyancy B and centre of gravity G of submerged bodies shown in Figure 3.20.

Fig. 3.20 (a) shows a balloon where the centre of buoyancy is always above the centre of gravity. A small angular displacement generates a restoring couple, between the buoyant force \( F_B \) and the weight \( W \), which brings the balloon back to its original position. This is an example of a stable equilibrium of a submerged body. In Fig. 3.20 (b) is shown a submerged body where the centre of buoyancy is below the centre of gravity. In this case, a small angular displacement generates a couple which further increases the displacement. This is a situation of unstable equilibrium. For a submerged, homogeneous spherical object shown in Fig...(c) the centre of gravity and the centre of buoyancy coincide and any angular
displacement does not result in development of a couple. In this case neutral equilibrium is said to occur.

The above considerations show that for completely submerged bodies the requirements for stability are:

(i) The centre of buoyancy and centre of gravity must lie on the same vertical line in the undisturbed position and

(ii) The centre of buoyancy must be located above the centre of gravity for stable equilibrium

3.9.3 Stability of Floating Bodies

For a floating body, the centre of buoyancy need not be located above the centre of gravity for stability. When a floating body which is partially submerged in a liquid is given a small angular displacement about a horizontal axis, the shape of the displaced volume of liquid changes and consequently the centre of buoyancy moves relative to the body. As a result, restoring couple can be generated and stable equilibrium achieved even
when the centre of gravity \( G \) of the body is above the centre of buoyancy \( B \).

Figure 3.21 may be used to illustrate the situation. Position (a) is the undisturbed position where the centre of buoyancy and the centre of gravity are on the same vertical. The weight \( W \) of the boat and the buoyant force \( F_B \) are equal, opposite and collinear. Hence the boat is in equilibrium.

![Figure 3.21 Stability of a floating body](image)

Position (b) shows the boat just as it has undergone through a small angular displacement \( \theta \). It is here assumed that the location of the centre of gravity \( G \) remains unchanged (this is true only for situation of unshifting cargo). In this position, the displaced volume on the right hand side increases and that on the left hand side decreases as a result of the displacement and the centre of buoyancy shifts to the right to a new position \( B' \). The buoyant force \( F_B \) (still equal to \( W \)) now acts vertically upwards through \( B' \) and the weight \( W \) acts downwards through \( G \). \( F_B \) and \( W \) now constitute a restoring counter-clockwise couple which brings the boat back to its original position. The line of action of \( F_B \) now intersects the axis \( BG \) at \( M \). This point \( M \) is known as the Metacentre. Thus, as long as \( M \) is above \( G \), a restoring couple will be generated and the floating object is in stable equilibrium. If \( M \) falls below \( G \), the generated couple will be an overturning couple and the equilibrium would be unstable.
Thus, for floating objects, stability would be achieved even when $B$ is below $G$ as long as the metacentre $M$ is above $G$. The special case where $G$ and $B$ coincide constitutes a situation of neutral equilibrium.

The distance of the metacentre $M$ above $G$ i.e. $MG$, is known as the metacentric height. It must be positive (i.e. $M$ must be above $G$) for stable equilibrium. For small values of heel angle $\theta$, the metacentric height is practically constant. The concept of metacentre and metacentric height is very useful in the design of ship profiles, barrages and caissons and the estimation of the metacentric height under various conditions of loading is important to ensure stability of the floating body.

**Metacentric Height**: An expression for the metacentric height may be obtained by considering the cross-section of a ship through its centre of gravity as shown Figure 3.22. The plan view at the water line is also shown.

In Figure 3.22, $AB$ is the original water line when the floating object was in the undisturbed, upright position with the centre of buoyancy $B$ and the centre of gravity $G$ in the same vertical axis of symmetry $BG$. $CD$ is the new water line after the floating object has experienced a small rotation through an angle $\theta$. As a result of the rotation, the triangular wedge $BOD$ on the right side has come out of the liquid while an identical wedge $AOC$ has gone inside. The total displaced volume does not change but its shape has changed and consequently the position of the centre of buoyancy shifts from $B$ to $B'$. The triangular wedges $AOC$ and $BOD$ correspond to a gain and a loss respectively in buoyant forces $\Delta F_B$.

The moment caused by these two forces is $\Delta F_B.S$ and has a clockwise sense. This must be equal to the opposite moment resulting from the shifting of the total buoyant force $F_B$ to $B'$. This moment is counterclockwise and is equal to $qg.V.\delta$, 89
Figure 3.22 Centre of buoyancy and metacentre of a floating body.

where $V$ is the total volume displaced by the floating object and $\rho$ is the liquid density.

Thus \[ \Delta F_B \cdot S = \rho g \cdot V \cdot \delta \]

Therefore \[ \delta = \frac{\Delta F_B \cdot S}{\rho g \cdot V} \]

Since \[ \delta = \overline{MB} \cdot \sin \theta, \]

\[ \overline{MB} = \frac{(\Delta F_B \cdot S)}{\rho g V \cdot \sin \theta} \]
The buoyancy force produced by wedge AOC (see Figure 3.22) can be estimated by considering a small prism of the wedge. Assume that the prism has a horizontal area $dA$ and is located at a distance $x$ from the axis of rotation $O$. The height of the prism is $x \cdot (\tan \theta)$. For small angle $\theta$ it may be approximated by $x \cdot \theta$. Thus the buoyancy force produced by the small prism is $\rho g x \theta \cdot ds$. The buoyancy force $\Delta F_B$ of the wedge AOC will be the sum of all these forces i.e.

$$\Delta F_B = \int \rho g x \cdot \theta \cdot dA$$

The moment produced by the couple is:

$$\Delta F_B \cdot S = \int_{-b/2}^{b/2} \rho g x \cdot \theta \cdot dA \cdot x = \rho g \theta \int_{-b/2}^{b/2} x^2 \cdot dA$$

or

$$\Delta F_B \cdot S = \rho g \theta \cdot I_{yy}$$

Where $I_{yy}$ is the second moment of area about axis $y-y$.

Substituting,

$$\overline{MB} = (\rho g \theta \cdot I_{yy}) / \rho g \cdot V \cdot \sin \theta = \frac{I_{yy} \cdot \theta}{V \cdot \sin \theta}$$

But limit $\theta / \sin \theta = 1$, $\theta \to 0$

Therefore,

$$\overline{MB} = \frac{I_{yy}}{V} \quad \text{3.21}$$
The Metacentric height \( \overline{MG} = \overline{MB} \pm GB \)

or \( \overline{MG} = \frac{I_{yy}}{V} \pm GB \)  \hspace{1cm} 3.22

Since the position of G and B is known from the sectional geometry or design data of the vessel, the distance GB can be determined. In Eqn. 3.22, the (+) sign is used when G is below B and the (-) sign used when G falls above B. If the value of \( \overline{MG} \) as determined above is positive, then the floating object is in stable equilibrium. If \( MG \) is negative, the floating object is unstable and if \( MG \) is zero, the object is in neutral equilibrium.

Example 3.15

A concrete block that has a total volume 1.5 m\(^3\) and specific gravity of 1.80 is tied to one end of a long hollow cylinder. The cylinder is 3 m long and has a diameter of 80 cm. When the assembly is floated in deep water, 15 cm of the cylinder remain above the water surface. Determine the weight of the cylinder.

Solution:

Referring to the following sketch: (Fig. E 3.15)

let \( W_c = \) Weight of the cylinder
\( W_b = \) Weight of the concrete block
\( F_b = \) Buoyant force of the assembly.
Volume of Water displaced = \( V \)

\[
V = \frac{\pi (0.8)^2}{4} (3 - 0.15) + 1.5 \text{ m}^3
\]

\[
= 1.433 + 1.5 = 2.933 \text{ m}^3
\]

Therefore,

\[
F_B = \gamma_w V = 9810 \times 2.933 = 28,772.7 \text{ N}
\]

\[
W_B = 1.80 \times 9810 \times 1.5 = 26,487.0 \text{ N}
\]

For equilibrium:

\[
W_c + W_B - F_B = 0
\]

Thus

\[
W_c = F_B - W_B
\]

\[
= 28772.7 - 26487.0 = 2285.7 \text{ N}
\]
Example 3.16

Two cubes of the same size, 1 m$^3$ each, one of specific gravity 0.80 and the other of specific gravity 1.1, are connected by a short wire and placed in water. What portion of the lighter cube is above the water surface? What is the tension in the wire?

Solution:

Referring to Figure E 3.16:

Let $\Delta V_1$ be the volume of submergence of the lighter cube.

$W_1 =$ Weight of lighter cube

$W_2 =$ Weight of heavier cube

$F_B =$ Total Buoyant force of the assembly

Then $F_B = (V_2 + \Delta V_1) \gamma_w = (1 + \Delta V_1) 9810 = 9810 + \Delta V_1 \cdot 9810$
For equilibrium,

\[ \sum F_y = 0 \]

i.e. \( F_B - W_1 - W_2 = 0 \)

\[ 9810 + 9810 \Delta V_1 - 0.8 \times 9810 \times 1 - 1.1 \times 9810 \times 1 = 0 \]

\[ 9810(1 + \Delta V_1 - 0.8 - 1.1) = 0 \]

\[ 1 + \Delta V_1 - 1.9 = 0 \]

Therefore, \( \Delta V_1 = 0.9 \)

Thus 0.1 of the volume or 10% of the lighter cube is above the water surface.

To determine the tension in the wire:
The buoyant force due to the heavier cube is:

\[ F_{B_1} = \gamma_w V_2 = 9810 \times 1 = 9810 \, N \]

Weight of heavier cube = \( W_2 \), 1.1 x 9810 x 1 = 10,791 N

Equilibrium of the heavier cube requires that:

\[ T + F_{B_2} - W_2 = 0, \quad \text{where } T = \text{tension in the wire.} \]

Thus \( T = W_2 - F_{B_1} = 10,791 - 9810 = 981 \, N \)
Example 3.17

A rectangular barge 20 m long has a 5 m wide cross-section. The water line is 1.5 m above the bottom of the barge when it floats in the upright position. If the centre of gravity is 1.8 m above the bottom, determine the metacentric height.

Solution:

Referring to Fig. E 3.17

\[ \frac{1.5}{2} = 0.75 \text{ m} \] above the bottom.

Since G is above B, the equation of the metacentric height will be:

\[ \bar{MG} = \frac{I_{yy}}{V} - GB \]

\[ V = \text{Volume displaced} = 20 \times 5 \times 1.5 = 150 \text{ m}^3 \]

\[ \frac{I_{yy}}{V} = \frac{bd^3}{12V} = \frac{20 \times 5^3}{12 \times 150} = 1.39 \text{ m} \]

\[ GB = 1.8 - 0.75 = 1.05 \text{ m} \]

Therefore, the metacentric height \( MG \) will be

\[ \bar{MG} = 1.39 - 1.05 = 0.34 \text{ m} \]
Example 3.18

A uniform wooden circular cylinder 400 mm in diameter and having a specific gravity of 0.6 is required to float in oil of specific gravity 0.8. Determine the maximum length of the cylinder in order that the cylinder may float vertically in the oil.

Solution:

$$\text{Weight of cylinder} = 0.6 \gamma_w \frac{\pi d^2}{4} \times l$$

$$\text{Weight of displaced volume of oil} = h \times 0.8 \gamma_w \frac{\pi d^2}{4}$$

Equating the two, depth of immersion $h$ will be:

$$h = \frac{0.6}{0.8} \times l = \frac{3}{8} l$$

$$OB = \frac{h}{2} = \frac{3}{8} l$$

$$OG = \frac{l}{2}$$

Therefore, $BG = OG - OB = \frac{l}{2} - \frac{3}{8} l = \frac{l}{8}$
Second moment of area of the circular section is

\[ I = \frac{\pi d^4}{64} \]

\[ I = \frac{\pi (400)^4}{64} = 400 \times 10^6 \pi \text{ mm}^4 \]

Volume of oil displaced = \( V \),

\[ V = \frac{\pi}{4} d^2 \cdot \frac{3l}{4} = \frac{3\pi d^2 l}{16} \text{ mm}^3 \]

\[ = \frac{3\pi (400)^2 l}{16} = 30 \times 10^3 \pi l \text{ mm}^3 \]

The metacentric height \( MG \) is given by:

\[ MG = \frac{I}{V} - BG \]

For the cylinder to float vertically in oil,

\[ MG \geq 0 \]

or \( \frac{I}{V} - BG \geq 0 \)

\[ \frac{I}{V} = \frac{400 \times 10^6 \pi}{30 \times 10^3 \pi l} = \frac{40}{3l} \times 10^3 \text{ mm} \]

\[ \frac{40}{3l} \times 10^3 \geq \frac{l}{8} \]

or \( l^2 \leq \frac{8 \times 40 \times 10^3}{3} \)

Therefore, \( l \leq 326.6 \text{ mm} \)
Example 3.19

A ship of 50 MN displacement floating in water has a weight of 100 kN moved 10 m across the deck causing a heel angle of 5°. Find the metacenteric height of the ship.

Solution:

Referring to Figure E 3.19

Moment causing the ship to heel = 100 x 10 = 1000 kN m.

\[ \text{= Moment due to shift of W from G to G'} \]

\[ \text{= W} \times \text{GG'} \]

But, \[ GG' = \frac{GM \sin \theta}{w} = \frac{1000}{w} = \frac{1000}{50 \times 10^3} = \frac{1}{50} \text{ m} \]

Hence, the Metacenteric height

\[ \bar{GM} = \frac{1}{50 \sin \theta} = \frac{1}{50 \sin 5°} = 0.23 \text{ m} \]
3.10 Relative Equilibrium of Liquids

If a liquid is contained in a vessel which is at rest, or moving with constant linear velocity, it is not affected by the motion of the vessel and the pressure distribution is hydrostatic. But if the container is given a continuous and constant linear acceleration or is rotated about a vertical axis with uniform angular velocity (resulting in a constant, inward acceleration), the liquid will eventually reach an equilibrium situation and move as a solid body with no relative motion between the fluid particles and the container. Such equilibrium of liquids is referred to as relative equilibrium of liquids. The two cases of practical interest are:

i) Uniform linear acceleration
ii) Uniform rotation about a vertical axis.

In both cases, since there is no relative motion between fluid particles, shear stress does not exist and the laws of fluid statics still apply, but in a modified form to allow for the effect of acceleration.

3.10.1 Uniform Linear Acceleration:

Consider the fluid element, shown in Fig. 3.23, in a vessel containing a liquid with density \( \rho \). Let the vessel be given a uniform linear acceleration with components \( a_x \), \( a_y \), and \( a_z \) along the \( x \), \( y \), and \( z \) directions respectively. Let the pressure at the centre of the element be \( P \) and pressure gradients \( \partial p/\partial x \), \( \partial p/\partial y \), \( \partial p/\partial z \) are assumed to exist in the \( x \), \( y \) and \( z \) directions respectively. The forces acting on the fluid element are shown.
Figure 3.23 Forces on fluid element under linear acceleration

Applying Newton's Second Law, the net force in the x-direction is:

\[
(p - \frac{\partial p}{\partial x} \cdot \frac{\Delta x}{2}) \Delta y \cdot \Delta z - (p + \frac{\partial p}{\partial x} \cdot \frac{\Delta x}{2}) \Delta y \cdot \Delta z = \rho \Delta x \cdot \Delta y \cdot \Delta z \cdot a_x,
\]

which reduces to:

\[
-\frac{\partial p}{\partial x} = \rho a_x \quad (3.23)
\]

In the y-direction, considering the weight of the fluid element, the net force will be

\[
(p - \frac{\partial p}{\partial y} \cdot \frac{\Delta y}{2}) \Delta x \cdot \Delta z - (p + \frac{\partial p}{\partial y} \cdot \frac{\Delta y}{2}) \Delta x \cdot \Delta z - \rho g \Delta x \Delta y \Delta z
\]

\[
= \rho a_y \Delta x \Delta y \Delta z
\]

which reduced to:

\[
-\frac{\partial p}{\partial y} = \rho (g + a_y) \quad (3.24)
\]
Similar considerations in the $Z$ direction lead to:

$$\frac{\partial p}{\partial z} = \rho a_z \tag{3.25}$$

Consider a vessel shown in Fig. 3.24 having uniform linear acceleration in the $x$-$y$ plane with components $a_x$ and $a_y$ in the $x$ and $y$ directions respectively.

![Figure 3.24 A vessel under uniform linear acceleration](image)

The total pressure differential is given by:

$$dp = \frac{\partial p}{\partial x} \cdot dx + \frac{\partial p}{\partial y} \cdot dy$$

On lines of constant pressure, the total pressure differential will be zero.

Thus:

$$\frac{\partial p}{\partial x} \cdot dx + \frac{\partial p}{\partial y} \cdot dy = 0$$

from which:

$$\frac{dy}{dx} = -\frac{\partial p/\partial x}{\partial p/\partial y} = -\frac{\rho a_x}{\rho (g + a_y)} = -\frac{a_x}{a_y + g} = \tan \theta \tag{3.26}$$
This shows that the lines of constant pressure have a constant slope of \( \tan \theta = -a_x/(a_y + g) \). Since the free surface is a line of constant pressure, the above conclusion also shows that lines of constant pressure are parallel to the free surface.

Once the position of the free surface is determined for a given acceleration, then the hydrostatic variation of pressure with depth applies as in fluid statics.

**Horizontal Acceleration:**

If a vessel containing a liquid moves with a constant linear horizontal acceleration \( a_x \), say in the positive \( x \) direction, then \( a_y = 0 \). Then the slope of the line of constant pressure i.e. the slope of the free surface will be

\[
\frac{dy}{dx} = \frac{-a_x}{g} = \tan \theta
\]

(3.27)

The variation of pressure with depth will be given by eqn.3.24, with \( ay = 0 \), as:

\[
\frac{dp}{dy} = -\rho g
\]

Integrating,

\[
p = -\rho gy + c
\]

Measuring the depth \( h \) from the free surface vertically down, \( y \) will be replaced by \((-h)\). Taking the free surface pressure as zero, the pressure \( p \) at a depth \( h \) from the free surface and at any section will be, as in hydrostatics,

\[
p = \rho gh
\]

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Example 3.20

A rectangular tank 5m long, 2m wide and 3m deep contains water filled to 1.5m depth. It is accelerated horizontally at 4 m/sec² in the direction of its length. Compute a) the total hydrostatic force acting on each side, (b) the force needed to impart the acceleration.

If the tank is completely filled with water and accelerated in the direction of its length at the rate of 2.5 m/sec², how many liters of water will be spilled?

Solution:

Referring to Figure E 3.20

\[
\tan \theta = \frac{a_x}{g} = \frac{4}{9.81} = 0.408
\]

Intercept \( AB = 2.5 \tan \theta = 1.02 \text{m.} = EF \)
Depth \( y_1 = 1.5 + 1.02 = 2.52 \text{m} \).

Depth \( y_2 = 1.5 - 1.02 = 0.48 \text{m} \).

\begin{align*}
\text{Hydrostatic Force on end AB} & = F_{AB} \\
F_{AB} & = \gamma hA \\
& = 9.81 \times \frac{2.52}{2} (2.52 \times 2) = 62.297 \text{ kN}
\end{align*}

\begin{align*}
\text{Hydrostatic Force on end ED} & = F_{ED} \\
F_{ED} & = \gamma hA \\
& = 9.81 \times \frac{0.48}{2} \times (0.48 \times 2) = 2.260 \text{ kN}
\end{align*}

\begin{align*}
\text{b) Force needed to impart acceleration} & = F_{AB} - F_{ED} \\
F_{AB} - F_{ED} & = 62.297 - 2.260 \\
& = 60.037 \text{ kN}
\end{align*}

Inertial force of accelerated mass = mass \times \text{acceleration} \\
= (1.5 \times 5 \times 2) \times 1000 \times 4 \\
= 60 \text{ kN}

Hence, difference between force on each end is equal to the inertial force.

When the tank is full,

\[
\tan \theta = \frac{a_x}{y} = \frac{2.5}{9.81} = 0.255
\]
Drop in water surface on front side = $5 \tan \theta = 1.275$ m

\[\therefore \text{volume of water spilled} = \frac{1}{2} \times 1.275 \times 5 \times 2\]

\[= 6.375 \text{ m}^3 = 6375 \text{ litres}\]

Example 3.21 A rectangular oil tanker 3 m wide, 2.0 m deep and 10 m long contains oil, $\rho = 800$ Kg/m$^3$, which stands at 1.0 m from the top of the tanker. Determine the maximum horizontal acceleration that can be given to the tanker without spilling the oil. If this tanker is closed and completely filled with oil and accelerated horizontally at 3 m/s$^2$ determine the total liquid thrust

i) on the front end   (ii) on the rear end and   (iii) on one of its longitudinal vertical sides.

Solution:

Referring to figure E 3.16

For maximum acceleration without spilling, the level drops by 1 m from the original level.
When the tanker is completely filled and closed, there will be pressure built up at the rear end equivalent to the virtual oil column \( h \) that would assume a slope of \( \frac{a_x}{g} \) (Fig. E 3.21(b))

\[
\tan \theta = \frac{a_x}{g} = \frac{1}{5}
\]

\[.\quad a_x = \frac{g}{5} = \frac{9.81}{5} = 1.962 \text{ m/s}^2
\]

i) Total thrust on front AB = \( \frac{1}{2} \rho g \cdot 2 \times 2 \times 3 = 58.86 \text{ kN} \)

ii) Total thrust on rear end CD

Virtual rise of oil level at rear end is \( h \)

\[ h = \tan \theta \cdot 10 = \frac{a_x}{g} \times 10 = \frac{3}{9.81} \times 10 = 3.06 \text{ m} \]

\[.\quad \text{Total thrust on CD} = \frac{\rho g(3.06) + \rho g(2 + 3.06)}{2} \cdot (2 \times 3)
\]

\[ = 239 \text{ kN} \]

iii) Total thrust on side ABCD = Volume of pressure prism which is equal to:

\[ \frac{1}{2} \rho g(2 \times 2 \times 10) + \frac{1}{2} \rho g(3.06)2 \times 10 \]

\[ = \rho g(20 = 30.6) = 496.4 \text{ kN} \]
Example 3.22 Calculate the slope of the free surface when an open container of liquid accelerates at 4.2 m/s²

i) In the horizontal direction

ii) Down a 30° inclined plane.

Solution:

i)

Slope of free surface is given by \( \frac{dy}{dx} \)

\[
\frac{dy}{dx} = \tan \theta = -\frac{a_x}{a_y + g}
\]

Here, \( a_x = 4.2 \text{ m/s}^2 \), \( a_y = 0 \)

\[
\therefore \theta = \tan^{-1} \left( \frac{4.2}{9.81} \right) = 23.18°
\]

ii)

When the acceleration down the 30° inclined plane is 4.2 m/s²,
\( a_x = -4.2 \cos 30° = -3.64 \text{ m/s}^2 \)
\( a_y = -4.2 \sin 30° = 2.1 \text{ m/s}^2 \)
Example 3.23  The U-tube shown in the figure below is filled with a liquid having a specific gravity of 2.40 and accelerated horizontally at 2.45 m/s². The leg of the U-tube is closed at the right end and open at the left end. Draw the imaginary free surface and determine the pressure at A. If the cross-sectional area of the tube is 6.28 cm², what volume of the liquid will be spilled?

Solution:
Referring to the above figure:

\[ \tan \theta = \frac{a_x}{g} = \frac{2.45}{9.81} = \frac{1}{4} = \frac{Y}{60} \]

\[ \therefore y = 60 \times \frac{1}{4} = 15 \text{ cm.} \]

\[ \therefore p_A = \gamma h = 2.4 \gamma \times 0.15 = 3.53 \text{ KN/m}^2 \]

Volume spilled = \( y \times \text{cross-sectional area of tube} \)

\[ = 15 \times 6.28 = 94.20 \text{ cm}^3 \]

**Vertical Acceleration**

If a liquid in a vessel is subjected to a constant vertical acceleration only, then \( a_x = 0 \). From Equation 3.23, \( \frac{\partial p}{\partial x} = 0 \) and from equation 3.26, \( \frac{\partial y}{\partial x} = 0 \). This means that the line of constant pressure is horizontal under vertical linear acceleration, i.e. the free surface remains horizontal. The pressure at any point in the liquid may be determined by integrating equation 3.24, i.e.,

\[ \partial p = \rho (g + a_y) dy \]

Or

\[ p = -\rho (g + a_y) y \]

Where \( y \) is measured vertically upwards in the positive \( y \) direction. To determine the pressure at any depth \( h \) below the free surface, \( y \) will be replaced by \((-h)\) and the pressure intensity is given by:
For a vessel that is accelerated vertically upwards, $a_y$ will be positive and for vertically downward acceleration, $a_y$ will be negative.

Example 3.24

A vertical hoist carries a square tank 2m x 2m containing water to the top of a construction scaffold with an acceleration of $2\text{m/s}^2$. If the water depth is 2m, calculate the total hydrostatic force on the bottom of the tank.

If this tank is lowered with an acceleration equal to that of gravity, what are the thrusts on the floor and sides of the tank?

Solution:

Since this is a case of vertical acceleration, the free surface and hence the lines of constant pressure remain horizontal.

Vertical upward acceleration = $a_y = 2\text{m/s}^2$

Pressure intensity at a depth $h = \rho(g + a_y)h$

$$= \rho g(1 + \frac{a_y}{g})h$$

$$= \rho g h (1 + \frac{2}{9.81})$$

$$= 1.204 \, gh \text{ kN/m}^2$$

\[\therefore\text{Total hydrostatic thrust on the floor}\]

$$= \text{intensity x area}$$

$$= 1.204 \times 9.81 \times 2 \times (2 \times 2) = 94.49 \text{ kN}$$

\[\]
Vertical downward acceleration = \(-9.81 \text{ m/s}^2\)
Pressure intensity at depth \(h\) = \(\rho gh(1-9.81/9.81) = 0\)

\[ : \text{There exists no hydrostatic force on the floor and on the side.} \]

3.10.2 Rotation about a vertical axis

When a vessel containing a liquid is rotated about a vertical axis at constant angular velocity, the liquid will, after a small adjustment period, rotate as solid body. Since there is no relative motion between adjacent layers of the liquid and between the liquid and the container, there are no shear stresses. Such a motion is called forced-vortex motion. As a result of the constant angular velocity \(\omega\), a constant, radially inward directed centripetal acceleration \(-\omega^2 \gamma\) acts on the fluid mass towards the axis of rotation. Consequently the pressure will vary in the radial direction because of the centrifugal effects.

In order to determine the variation of pressure in the radial direction, consider a small element of fluid of length \(dr\) and cross sectional area \(dA\) at radial distance \(r\) in liquid mass which is contained in a cylinder of internal radius \(r_o\) shown in figure 3.25. The cylinder is rotated at constant angular velocity \(\omega \text{ rad/s}\) about its vertical axis.

The mass of the element is \(\rho \cdot dr \cdot dA\). This mass is subjected to a radially inward acceleration \(-\omega^2 \gamma\).

Newton's Second Law applied to the element will be:

\[ p \cdot dA - (p - \frac{\partial p}{\partial r} \cdot dr) \cdot dA = \rho dr \cdot dA(-\omega^2 r) \]
Figure 3.25 Rotation with constant angular velocity

Which when simplified reduced to:

\[ \frac{\partial P}{\partial z} = \rho \omega^2 r \quad (3.29) \]

The variation of pressure with depth will be obtained by considering the forces in the vertical direction with gravitational acceleration acting on the fluid element. This leads to the variation of pressure with depth to be the same as when the liquid is at rest i.e.

\[ \frac{\partial P}{\partial z} = \rho g \quad (3.30) \]

For the surface of constant pressure, the total pressure differential will be zero.

i.e \[ dp = \frac{\partial P}{\partial r} . dr + \frac{\partial P}{\partial z} . dz = 0 \]
or, \[ 0 = \rho \omega^2 r \, dr - \rho g dz \]

Leading to:

\[
\frac{dz}{dr} = \frac{\omega^2 r}{g} \quad (3.31)
\]

Integrating the above,

\[
z = \frac{\omega^2 r^2}{2g} + c \quad (3.32)
\]

This shows that the constant pressure lines are parabolic. Considering the free surface which is constant pressure surface,

\[ r = 0 \text{ at } z = 0, \] which make \( c = A \) in equation 3.32

Thus:

\[ Z - A = \frac{\omega^2 r^2}{2g} \]

At the container's wall, \( r = r_0 \) and \( z = z_0 \). Therefore,

\[
z_0 - A = h_0 = \frac{\omega^2 r_0^2}{2g} \quad (3.33)
\]

Equation 3.33 shows that for a circular cylinder rotating about its axis, the rise of liquid along the wall from the vertex is \( \omega^2 r_0^2/2g \).
Consider a cylindrical tank partially filled with a liquid and rotated about its vertical axis at constant angular velocity \( \omega \) rad/sec so that no liquid is spilled as shown in Figure 3.26.

![Figure 3.26](image)

The shaded volume \( \text{AED} \) = paraboloid of revolution
= volume of the empty space \( \text{ABCD} \)

Paraboloid of revolution = \( \frac{1}{2} \) (volume of circumscribing cylinder)

\[ \therefore \text{Volume of empty space } \text{ABCD} = \frac{1}{2} (\pi r_0^2 h_0) \]

\[ i.e. \quad \pi r_0^2 h_s = \frac{1}{2} \pi r_0^2 h_o \]

\[ \therefore h_s = \frac{1}{2} h_o = h_s \]

This shows that during rotation about a vertical axis at constant angular velocity, the liquid rises along the walls the same amount above the rest level as the centre drops at the axis below the rest level.
Example 3.25

An open cylindrical tank, 2 m high and 1 m in diameter, contains 1.5 m depth of water. If the cylinder rotates about its vertical geometric axis,

a) What is the maximum constant angular velocity that can be attained without spilling any water?

b) What is the pressure intensity at the centre and corner of the bottom of the tank i.e. at C and D (fig. E 3.25) when the angular velocity is

Solution:

\[\text{Figure E 3.25}\]

a) If no liquid is to be spilled, the maximum angular velocity \( \omega \) will have such a magnitude that will enable the liquid to rise to level B at the wall of the cylinder.

Under this condition,

\[
\text{Volume of paraboloid of revolution} = \text{Volume of original empty space}
\]
\[
\frac{1}{2} (\pi \times 0.5^2) (y_1 + 0.5) = \pi \times 0.5^2 \times 0.5
\]

From the above, \( y_1 = 0.5 \) m

Thus:

\[
h_o = 1 \text{m} = \frac{\omega^2 \times 0.5^2}{2 \times 9.81}
\]

i.e \( \omega = [(2 \times 9.81)/0.5^2]^{1/2} = 8.86 \text{ rad/s} \)

b) For \( \omega = 8 \text{ rad/s} \),

\[
h_o = \frac{8^2 \times 0.5^2}{2 \times 9.81} = 0.816 \text{m}
\]

S drops by 1/2 \( h_o = 0.405 \) from level A-A.
Thus: at C, depth from free surface = 1.5 - 0.408 = 1.092 m
at D, depth from free surface = 1.5 + 0.408 = 1.908 m

\[
P_C = \rho gh_C = 9810 \times 1.092 = 10,713 \text{ N/m}^2 = 10.713 \text{ kPa}
\]

\[
P_D = \rho gh_D = 9810 \times 1.908 = 18,717 \text{ N/m}^2 = 18.72 \text{ kPa}
\]

Example 3.26

If the tank in the above example is closed at the top and the air subjected to a pressure of 1.07 bar (= 107 KN/m\(^2\)), determine the pressures at points C and D when the angular velocity is 115 rpm.

Solution:

Referring to Figure E 3.26:
Since there is no change in the volume of air within the tank, volume above level A-A = volume of empty space = volume of paraboloid of revolution
Substituting the value of $r_2^2$ from (2) in (1):

$$\frac{7.39}{2} \cdot r_2^4 = 0.125$$

$$\therefore r_2 = 0.43 \text{ m}$$
and \[ y_2 = 7.39 \times (0.43)^2 = 1.36 \text{ m}. \]

Thus, \( S \) is located \((2-1.36) = 0.64 \text{ m above } c\)

\[ y_1 = \frac{\omega^2 \times r_1^2}{2g} = \frac{12.04^2 \times 0.5^2}{2 \times 9.81} = 1.847 \text{ m}. \]

\[ \text{:. Pressure head at } D = 0.64 + 1.847 = 2.487 \text{ m}. \]

\[ \text{:. Pressure at } C = P_c = P_{\text{air}} + \rho gh_c \]
\[ = 1.07 \times 10^5 + 9810 \times 0.64 \]
\[ = (1.07 + 0.063) \times 10^5 \text{ P}_a \]
\[ = 1.133 \times 10^5 \text{ P}_a \]

\[ \text{Pressure at } D = P_D = P_{\text{air}} + \rho gh_D \]
\[ = 1.07 \times 10^5 + 9810 \times 2.487 \]
\[ = (1.07 + 0.244) \times 10^5 \]
\[ = 1.314 \times 10^5 \text{ P}_a \]

Example 3.27

A closed cylindrical vessel 1 m in diameter and 1.8 m high contains water to a depth of 1.3 m. If the vessel is rotated at 18 rad/s, what is the radius of the circle that will be uncovered at the bottom of the vessel?

Solution:

Referring to Figure E 3.27

assume that the vertex \( s \) of the paraboloid is at a distance \( h \) m below the bottom of the vessel.
Then: 

\[ h_2 = \frac{\omega^2 r_1^2}{2 \times 9.81} = \frac{18^2 \times r_1^2}{19.62} = 16.51 \times r_1^2 \]

from which, 

\[ r_1^2 = \frac{h_1}{16.51} \]  

(1)

\[ h_2 = (1.8 + h_1) = \frac{18^2 \times r_2^2}{19.62} = 16.51 \times r_2^2 \]

from which, 

\[ r_2^2 = \frac{(1.8 + h_1)}{16.51} \]  

(2)

\[ h_3 = \frac{18^2 \times r_3^2}{19.62} = 16.51 \times 0.5^2 = 4.13 \text{ m.} \]
Volume of water in the vessel = Volume of cylinder - (volume of paraboloid ABS - volume of paraboloid DSC)

\[
\frac{\pi}{4} \times 1^2 \times 1.3 = \frac{\pi}{4} \times 1^2 \times 1.8 - \left( \frac{1}{2} \pi r_2^2 h_2 - \frac{1}{2} \pi r_1^2 h_1 \right)
\]

\[
= \frac{\pi}{4} \left[ 1.8 - 2r_2^2 h_2 + 2r_1^2 h_1 \right]
\]

or

\[1.3 = 1.8 - 2r_2^2 h_2 + 2r_1^2 h_1\]

Substituting the values of \(r_1^2\) and \(r_2^2\) from (1) and (2) above,

\[1.3 = 1.8 - 2(1.8 + h_1)(1.8 + h_1)/16.51 + 2h_1^2/16.51\]

\[-0.5 \times 16.51 = -2(3.24 + 3.6h_1 + h_1^2) + 2h_1^2\]

\[-8.255 = -6.48 - 7.2h_1\]

\[\therefore h_1 = 0.247 \text{ m.}\]

and \(r_1 = (0.247/16.5)\sqrt{2} = 0.122 \text{ m} = 12.2 \text{ cm}\)

Example 3.28

The U-tube in Example 3.23 is rotated about a vertical axis 15 cm to the right of A at such a speed that the pressure at A is zero gauge. What is the rotational speed?
Solution:

Referring to Figure E 3.28

If the pressure at A is to be zero gauge i.e. atmospheric, then the paraboloid of revolution which passes through B must also pass through A. The vertex will be at S.

Thus: 

\[ y_2 = y_1 + 0.30 = \frac{\omega^2 r_2^2}{2g} = \frac{\omega^2(0.75)^2}{2g} \]  \hspace{1cm} (1)

\[ y_1 = \frac{\omega^2 r_1^2}{2g} = \frac{\omega^2(0.15)^2}{2g} \]  \hspace{1cm} (2)
Substituting value of \(y_1\) from (2) into (1):

\[
\frac{\omega^2(0.5625)}{2g} = \frac{\omega^2 \times (0.0225)}{2g} + 0.3
\]

\[
\omega^2(0.5625 - 0.0225) = 0.3 \times 2g
\]

\[
\omega = [(0.3 \times 19.62)/0.54]^{1/2} = 3.30 \text{ rad/s}
\]

\[
= 31.5 \text{ rPm}
\]

Exercise Problems

3.1 What will be (a) the gauge pressure, (b) the absolute pressure of water at a depth of 20 m below the free surface. Assume the density of water to be 1000 kg/m³ and the atmospheric pressure 101 kN/m². (Ans. 196.2 kN/m², 297.2 kN/m²)

3.2 Calculate the pressure in the ocean at a depth of 2000 m assuming that salt water is (a) incompressible with a constant density of 1002 kg/m³, (b) compressible with a bulk modulus of 2.05 GN/m² and a density at the surface of 1002 kg/m³.

3.3 An inverted U-tube is used to measure pressure difference between A and B (Fig. P.33). If the top space in the tube is filled with air, what is the difference in pressure between A and B, when (a) water (b) oil of relative density 0.65 flows through the pipes.

3.4 Find the value of \(h\) in metres in Fig. P 3.4 when the air pressure above the surface is 3.5 m of water below atmospheric (The manometric liquid has \(s = 2.5\))
3.5 At what height $H$ of water will the conical valve in Fig. P 3.5 start to leak? The valve weighs 2.256 kN and assume the pulley to be frictionless. (Ans. $H = 1.325$ m)
3.6 A 6 m x 2 m rectangular gate is hinged at the base and is inclined at an angle of $60^\circ$ (Fig P 3.6). If $W = 39.2\, \text{kN}$ acting at angle of $90^\circ$ to the gate find the depth of the water when the gate begins to fall. Neglect the weight of the gate and the friction of the pulley.

3.7 A gate consists of a quadrant of a circle of radius 1.5m pivoted at 0 (Fig. P 3.7). The centre of gravity of the
gate is at G. Calculate the magnitude and direction of the resultant force on the gate due to the water and the turning moment required to open the gate. The width of the gate is 3 m and it has a mass of 5000 kg.

(Ans. 61.6 kN, 57° 28', 29.417 kN m.)

Fig. P 3.7

3.8 A submarine weighing 3.924 MN has an enclosed volume of 800 m³. What volume of water should be taken in to submerge the vessel?

3.9 An open steel tank having a 3.3 m x 3.3 m plan section and a draft of 1.3 m has its centre of gravity at the water line. The tank has to be delivered by towing after fabrication to its final location. Determine whether it will float stably without adding ballast.

(Ans: Stable)

3.10 The open rectangular tank shown in Fig p 3.10 is 5 m wide, 6 m deep and 10 m long. It is filled to a depth of 4 m with water. If the tank is accelerated horizontally at 1/2g, calculate

a) The volume of water spilled (if any)
b) The force on the back and fron end
3.11 In figure P 3.11, calculate the minimum volume of the concrete block \( (y_c = 22.5633 \text{ KN/m}^3) \) which will hold the circular gate AB in place. The block is submerged in water. The pulley is frictionless. (Ans. 1.264 m\(^3\))

3.12 In figure P 3.12, \( a_x = 2.45 \text{ m/s}^2, a_y = 4.90 \text{ m/s}^2 \). Determine
a) The angle which the free surface makes with the horizontal
b) The pressure at B and C in N/m\(^2\)

3.13 An open cylindrical tank 1.2 m in diameter and 1.8 m deep is filled with water and rotated about its axis at 60 rpm. How much liquid is spilled and how deep is the water at the axis? (Ans. 0.43 m\(^3\), 1.1 m)

3.14 At what speed should the tank in problem 3.13 be rotated in order that the center of the bottom of the tank have zero depth of water?
<table>
<thead>
<tr>
<th>Shape</th>
<th>Area</th>
<th>Centroid</th>
<th>$I_{cc}$</th>
<th>$I_{xx}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangle</td>
<td>$b \cdot h$</td>
<td>$\bar{x} = \frac{1}{2} b$</td>
<td>$\frac{1}{2} bh^3$</td>
<td>$\frac{bh^3}{3}$</td>
</tr>
<tr>
<td>Triangle</td>
<td>$\frac{1}{2} b \cdot h$</td>
<td>$\bar{x} = \frac{b + c}{3}$</td>
<td>$\frac{1}{36} bh^3$</td>
<td>$\frac{bh^3}{12}$</td>
</tr>
<tr>
<td>Circle</td>
<td>$\frac{\pi d^2}{4}$</td>
<td>$\bar{x} = \frac{d}{2}$</td>
<td>$\frac{\pi d^4}{64}$</td>
<td>$\frac{d}{2}$</td>
</tr>
</tbody>
</table>

Table 3.1: Surface Area, Centroid and Second Moment of Area of Some Simple Geometrical Shapes
<table>
<thead>
<tr>
<th>Shape</th>
<th>Area</th>
<th>Centroid</th>
<th>$I_{xx}$</th>
<th>$I_{yy}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trapezium</td>
<td>$(a + b)h \over 2$</td>
<td>$\bar{y} = h(2a + b) \over 3(a + b)$</td>
<td>$h^3(a^3 + 4ab + b^2) \over 36(a + b)$</td>
<td>$(3a + b)h^3 \over 12$</td>
</tr>
<tr>
<td>Semicircle</td>
<td>$\frac{1}{2} \pi r^2$</td>
<td>$\bar{y} = 4r \over 3\pi$</td>
<td>0.115</td>
<td></td>
</tr>
<tr>
<td>Ellipse</td>
<td>$\pi bh$</td>
<td>$\bar{x} = b$</td>
<td>$\pi \over 4 bh^3$</td>
<td></td>
</tr>
<tr>
<td>Parabola</td>
<td>$\frac{2}{3} bh$</td>
<td>$\bar{y} = \frac{2}{5} h$</td>
<td>$\frac{8}{175} bh^3$</td>
<td></td>
</tr>
</tbody>
</table>
CHAPTER 4

KINEMATICS OF FLOW

4.1 Introduction:

In hydrostatics one deals with liquids at rest in which there is no relative motion between fluid particles and therefore no shear stresses exist. Since no friction is involved the fluid may be assumed to be either ideal or real.

In fluid flow problems one often refers to the flow of an ideal fluid. An ideal fluid flow as the name implies, is an idealized situation in which the fluid is assumed to have no viscosity and therefore no shear stresses exist. Boundary effects are ignored and velocities assumed to be uniform. Such simplification is sometimes useful in solving some engineering problems. When dealing with real fluid flow, however, the effects of viscosity are introduced thus leading to considerations of the developed shear stresses between neighbouring fluid layers that are moving at different velocities. The flow picture thus becomes complex and can not be easily formulated mathematically as in the idealized situation. It requires the combination of mathematical theory with experiments. Study of fluids in motion thus requires consideration of fluid properties (such as specific weight, viscosity etc), kinematics and force and energy relationships. Kinematics deals only with the geometry of motion i.e. space time relationships of fluids only without regard to the forces causing the motion.

4.2 Velocity Field

There are two methods or frames of reference by which the motion of a fluid can be described:

(i) The lagrangian Method and

(ii) The Eulerian Method
The Lagrangian Method: In this method the observer focusses his attention on a single fluid particle during its motion through space to find out the path it traces and to describe its characteristics such as velocity, acceleration, density etc as it moves in the flow field with the passage of time. In the cartesian coordinate system, the position of a fluid particle in space \((x,y,z)\) is expressed with respect to a coordinate system \((a,b,c)\) at time \(t_0\). Thus, \(a,b,c\) and \(t_0\) are independent variables while \(x,y,z\) are dependent variables in this method. If \(u\) is the velocity of the fluid particle at time \(t\), then its position \(x = a + ut\) where \(a\) is its x coordinate at time \(t_0\). Thus the position of the particle will be:

\[
\begin{align*}
    x &= f_1(a,b,c,t) \\
    y &= f_2(a,b,c,t) \\
    z &= f_3(a,b,c,t)
\end{align*}
\]

(4.1)

The corresponding velocities \(u\), \(v\) and \(w\) and accelerations \(a_x\), \(a_y\) and \(a_z\) in the \(x, y,\) and \(z\) directions will be:

\[
\begin{align*}
    u &= \frac{\partial x}{\partial t} \\
    a_x &= \frac{\partial^2 x}{\partial t^2} \\
    v &= \frac{\partial y}{\partial t} \\
    a_y &= \frac{\partial^2 y}{\partial t^2} \\
    z &= \frac{\partial z}{\partial t} \\
    a_z &= \frac{\partial^2 z}{\partial t^2}
\end{align*}
\]

(4.2)

The Lagrangian Method of analysis is difficult in fluid mechanics since it is not easy to identify a fluid particle and because every particle has a random motion.

The Eulerian Method: In this method the observer's concern is to know what happens at any given point in the space which is filled with a fluid. One is interested in what the velocities, accelerations, pressures etc are at various points in the flow.
field at any given time. This method is extensively used in fluid mechanics because of its simplicity and due to the fact that one is more interested in flow parameters at different points in a flow and not in what happens to individual fluid particles. The position of a particle in this method is expressed with respect to a fixed coordinate system $x, y, z$ at a given time $t$. The Eulerian velocity field is thus given by:

$$
\begin{align*}
    u &= f_1(x, y, z, t) \\
    v &= f_2(x, y, z, t) \\
    w &= f_3(x, y, z, t)
\end{align*}
$$

Since equation (4.2) describes the motion of a single fluid particle, the relationship between the Lagrangian and the Eulerian equations will be:

$$
\begin{align*}
    \frac{dx}{dt} &= u(x, y, z, t) \\
    \frac{dy}{dt} &= v(x, y, z, t) \\
    \frac{dz}{dt} &= w(x, y, z, t)
\end{align*}
$$

The integration of Equations (4-4) leads to the Lagrangian equations (4-1) with the initial conditions $x = x_0 = a; y = y_0 = b; z = z_0 = c$ and $t = t_0$. Hence, the Lagrangian Method can be derived from the Eulerian Method.

4.3 Velocity and Acceleration:

The motion of fluid particles in a particular flow phenomenon is expressed in terms of a vector quantity known as velocity. In figure (4-1), if $\Delta s$ is the distance travelled by a fluid
particle in time $\Delta t$, then $\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$ is called the velocity $V$, and it is tangential to the path $s$.

![Tangential Velocity Diagram](image)

**Figure 4.1** Tangential velocity

Since the distance $\Delta s$ can be resolved in general into distances $\Delta x$, $\Delta y$ and $\Delta z$ in the $x,y$ and $z$ directions respectively, the velocity vector $V$, in turn can be resolved into components $u$, $v$ and $w$ in the $x,y$ and $z$ directions respectively. The magnitudes of each of the component velocities will in general depend on the location of the point under consideration i.e on $x,y$ and $z$ and also on time $t$ depending upon the type of flow. Thus the velocity $V$, and components can be written in the following functional form:

$$V_s = f_1(x,y,z,t)$$
$$u = f_2(x,y,z,t)$$
$$v = f_3(x,y,z,t)$$
$$w = f_4(x,y,z,t)$$

Consider now a particle of fluid moving from $A$ to $B$ on a streamline as shown in figure 4.2

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The velocity of the particle may change for two reasons: At a particular instance, the velocity at A may be different from the velocity at B and also during the period the given particle moves from A to B, the velocity at B might change. Thus the total change in the velocity of the particle, \( dv_s \), will be the sum of its change due to change in position and its change due to passage of time interval \( dt \).

\[
\begin{align*}
\text{i.e.} \quad dv_s &= \frac{\partial v_s}{\partial s} \, ds + \frac{\partial v_s}{\partial t} \cdot dt.
\end{align*}
\]

The tangential acceleration \( a_s \), in the flow direction will be:

\[
\begin{align*}
a_s &= \frac{dv_s}{dt} = \frac{\partial v_s}{\partial s} \cdot \frac{ds}{dt} + \frac{\partial v_s}{\partial t} \cdot \frac{dt}{dt} \\
&= \frac{\partial v_s}{\partial s} \cdot \frac{ds}{dt} + \frac{\partial v_s}{\partial t} \\
&= V_s \frac{\partial v_s}{\partial s} + \frac{\partial v_s}{\partial t} \\
\end{align*}
\]

Thus \( a_s = V_s \frac{\partial v_s}{\partial s} + \frac{\partial v_s}{\partial t} \) \hspace{1cm} (4.6)
where: \( a_s = \) local tangential acceleration
\[
\frac{\partial v_s}{\partial t} = \text{the local or temporal component}
\]
= the rate of change of velocity with respect to time at a particular point

\( v_s \frac{\partial v_s}{\partial t} = \text{the convective component}
\]
= the rate of change of velocity due to the particle's change of position

The normal acceleration, which is the result of change in direction of velocity may be obtained by considering the curved stream line and the velocity vector diagram shown in Figure (4-3).

\[ V \]
\[ D \]
\[ A \]
\[ \Delta s \]
\[ \Delta V_n \]
\[ \Delta V \]
\[ R \]
\[ \theta \]

\[ V \]
\[ V + \Delta V \]
\[ V + \Delta V \]

Figure 4.3

\( V \) is the tangential velocity at A
\( V + \Delta v \) is the tangential velocity at D at a distance \( \Delta s \) from A.
\( \Delta V_n \) is the velocity in the normal direction at A.
\( R \) is the radius of curvature of the stream line.
From the velocity vector diagram,

\[ \Delta V_n = V d\theta = V \cdot \frac{\Delta s}{R} \]

The convective acceleration \( a_n \) in the normal direction is given by:

\[ a_n = \frac{\Delta V_n}{\Delta t} = \frac{\Delta V_n}{\Delta s} \cdot \frac{\Delta s}{\Delta t} = \frac{V}{R} \cdot V = \frac{v^2}{R} \]

The normal velocity can also change with time. Hence, the temporal component of the normal acceleration will be \( \frac{\partial V_n}{\partial t} \).

Therefore, the normal acceleration with its temporal and convective components will be:

\[ a_n = \frac{\partial V_n}{\partial t} + \frac{v^2}{R} \] \hspace{1cm} (4.7)

4.4 Pathline, Streak line, Streamline and Steam tube

Pathline: If an individual particle of fluid is coloured, it will describe a pathline which is the trace showing the position at successive intervals of times of a particle which started from a given point.

Streakline or Filament line: If, instead of colouring an individual particle, the flow pattern is made visible by injecting a stream of dye into a liquid, or smoke into a gas,
the result will be a streakline or filament line, which gives an instantaneous picture of the positions of all the particles which have passed through the particular point at which the dye is being injected. Since the flow pattern may vary from moment to moment, a streak line will not necessarily be the same as a pathline.

**Streamline:** A streamline is defined as an imaginary line drawn through a flow field such that the tangent to the line at any point on the line indicates the direction of the velocity vector at that instant. A streamline thus gives a picture of the average direction of flow in a flow field.

Since the velocity vector at any point on a streamline is tangential it will not have a component normal to the streamline. Hence there can not be any flow across a streamline. Thus the flow between any two streamlines remains constant. A smooth flow boundary can also be considered as a streamline. If conditions are steady and the flow pattern does not change from moment to moment, pathlines and streamlines are identical.

Consider a streamline shown in figure (4-4). The velocity vector $V_s$ at point $P(x,y)$ has components $u$ and $v$ in the $x$ and $y$ directions respectively.

![Figure 4.4 Streamline](image-url)
Taking $\theta$ as the angle between $V$, and the $x$ axis, 

$$\tan \theta = \frac{dy}{dx} = \frac{v}{u}$$

Thus, the equation of a streamline in two dimensional flow at any instant $t_o$ is:

$$\frac{dx}{u} = \frac{dy}{v} \quad \text{Where} \quad u = f_1(x, y, t_o) \quad v = f_2(x, y, t_o)$$

Example 4.1 If $u = +x$, and $v = 2y$, find the equation of the streamline through $(1,1)$.

$$\frac{dx}{u} = \frac{dy}{v}$$

or 

$$\frac{dx}{x} = \frac{dy}{2y}$$

$$\ln x = \frac{1}{2} \ln y + C$$

at $x = 1$ and $y = 1$, $C = 0$

$$\therefore \quad x = \sqrt{y} \text{ is the equation of the streamline}$$

Streamtube: If a series of streamlines are drawn through every point on the perimeter of a small area of a stream crosssection, they will form a streamtube. Since there is no flow across a streamline, there will not be flow across a streamtube and the fluid inside a streamtube cannot escape through its walls. The flow thus behaves as if it were contained in an imaginary pipe. The concept of a streamtube is useful in dealing with the flow of fluids since it allows elements of the fluid to be isolated for analysis.
4.5 Classification of flows

In a general flow field, velocity, pressure, density etc. can vary from place to place or can change with respect to time or both variations can occur simultaneously. It is convenient to classify flows on the basis of change in velocity only. Accordingly, a flow may be classified as steady or unsteady depending upon whether the velocity at a point varies with time or not and as uniform or non-uniform depending upon whether the velocity at different points on a streamline in a flow field at an instant is the same or not.

When any of the flow parameters at a point do not change with time, the flow is said to be steady. Variations of any of the flow parameters with time at a point would cause the flow unsteady.

Steady and unsteady conditions refer only to average temporal velocity in a flow field and turbulent fluctuations are not considered.

Velocities in a flow field depend on the geometry of the boundary. Consider the three situations in Figure 4-6.
If the rate of flow does not vary with time i.e the flow is steady, then the average velocities at any two sections 1-1 and 2-2 in Figure 4-6(a) would be the same. In the expanding pipe of Figure 4-6(b), the velocities at sections 1-1 and 2-2 are different since the cross-sectional areas vary. In the bend of constant diameter shown in Figure 4-6(c), eventhough the magnitudes of the velocities at sections 1-1 and 2-2 are the same, their directions are different and hence there is variation in velocity.

A flow is considred uniform if velocities at different points in a steamline (or average velocities at different sections in a conduit) in a flow field at an instant are the same both in magnitude and direction. If there is variation either in magnitude or direction or both, then the flow is said to be non-uniform. Thus flow in the straight pipe of uniform diameter is classified as uniform while those in the expanding pipe and the bend are non-uniform.

Considering both temporal and spatial (convective) variations in the flow parameters, the following four combinations of flow are possible:

i) Steady uniform flow - flow through a uniform diameter pipe with a constant rate of flow.
ii) Steady non-uniform flow - flow through a straight pipe with changing diameter (expanding or reducing) and a bend with uniform or non-uniform diameter at constant rate of flow.

iii) Unsteady uniform flow - flow through a uniform diameter pipe at changing rates of flow.

iv) Unsteady non-uniform flow - flow as in (ii) but with changing rate of flow.

4.6 One, Two and Three-Dimensional Flows

When the velocity components transverse to the main flow direction is neglected and only average conditions of flow are considered at a section then the flow is said to be one-dimensional. The assumption of one dimensional flow can be made where there is no wide variation of cross-section, where stream lines are not highly curvilinear and where the velocity variation across a section is not appreciable. Many engineering problems such as flows through a pipe and open channel flows are handled by one dimensional analysis by taking average values of the flow characteristics at sections.

In actual flows of real fluids, the presence of fluid viscosity and the no slip condition at the boundary require that the velocity vary from zero at the boundary to a maximum value somewhere in the flow field depending upon the boundary conditions. Such a flow where the velocity vector is a function of two co-ordinates is known as two-dimensional. Flow past a wide flat plate or over a long weir can be considered two-dimensional.

Figure 4.7 illustrates one and two-dimensional flows. In figure 4.7(a) there is a velocity variation only in the flowdirection and velocities are consatant at each of the...
sections 1-1 and 2-2. In figure 4.7(b) velocity variations occur in both x and y directions.

![Figure 4.7 One and two dimensional flow](image)

Three dimensional flow is the most general type of flow in which the velocity vector varies in the three coordinate directions x, y and z and is generally complex.

Thus in terms of the velocity vector \( V \), the following apply:

<table>
<thead>
<tr>
<th></th>
<th>Unsteady</th>
<th>Steady</th>
</tr>
</thead>
<tbody>
<tr>
<td>One-dimensional flow</td>
<td>( v = f(x,t) )</td>
<td>( v = f(x) )</td>
</tr>
<tr>
<td>Two-dimensional flow</td>
<td>( v = f(x,y,t) )</td>
<td>( v = f(x,y) )</td>
</tr>
<tr>
<td>Three-dimensional flow</td>
<td>( v = f(x,y,z,t) )</td>
<td>( v = f(x,y,z) )</td>
</tr>
</tbody>
</table>

4.7 Discharge and Mean Velocity

The total quantity of fluid flowing in unit time past any particular cross-section of a stream is called the discharge or flow at that section. It can be measured either in terms
of mass, in which case it is referred to as the mass rate of flow \( \dot{m} \) (eg. in kg/s), or it can be measured in terms of volume, when it is known as the volumetric rate of flow \( Q \) (eg. in \( m^3/s \)).

In many problems, the variation of velocity over the cross-section can be ignored and the velocity is assumed to be constant and equal to the mean velocity \( \overline{V} \). If the cross-sectional area normal to the direction of flow is \( A \), the volume passing the cross-section in unit time would be \( A\overline{V} \). Thus:

\[
Q = A\overline{V}
\]

or The mean velocity \( \overline{V} = \frac{Q}{A} \)

In a real fluid flow, the velocity adjacent to a solid boundary will be zero. The velocity profile across a section of a pipe for laminar and turbulent flows are as shown in Figure 4.8.

If \( u \) is the velocity at any radius \( r \), the flow \( dQ \) through an annular element of radius \( r \) and thickness \( dr \) will be:

\[
dQ = \text{Area of element} \times \text{velocity} = 2\pi r dr \cdot u
\]

Hence, \( \int dQ = \int_0^R 2\pi r u dr \)

or \( Q = 2\pi \int_0^R ur dr \)
The above integral can be evaluated if the relation between $u$ and $r$ can be established.

In general, if $u$ is the velocity at any arbitrary location in the profile and $A$ the total flow area, then the average velocity is given by:

$$
\overline{V} = \frac{\int_A u \, dA}{A}
$$

(4.8)

Example 4.1

The velocity distribution for laminar flow between parallel plates is given by: $u = K(Dy - y^2)$ where $u$ is the velocity at distance $y$ from the bottom plate, $D$ is the distance between the plates and $K$ is a constant. Determine the average velocity of flow.
Solution:

Taking a unit width of the plates, and letting the average velocity be \( \bar{V} \),

\[
\bar{V} D.1 = \int_0^D u \, dy.1
\]

or

\[
\bar{V} = \frac{1}{D} \int_0^D k(Dy^2 - y^3) \, dy
\]

\[
= \frac{K}{D} \left[ \frac{Dy^2}{2} - \frac{y^3}{3} \right]_0^D = \frac{kD^2}{6}
\]

4.8 Continuity Equation

The equation of continuity is the mathematical expression for the principle of conservation of mass flow.

4.8.1 One Dimension, Steady Flow:

Consider a steam tube through which passes a steady flow of fluid.

Figure 4.9
At Section (1): 

- \( dA_1 = \text{cross-sectional area of stream tube} \)
- \( v_1 = \text{avg velocity through stream tube} \)
- \( \rho_1 = \text{fluid density} \)

At Section (2): 

- \( dA_2 = \text{Cross-sectional area of stream tube} \)
- \( v_2 = \text{avg. velocity through stream tube} \)
- \( \rho_2 = \text{fluid density} \)

Mass rate of flow through \( dA_1 = \rho_1 dA_1 v_1 \)

For the entire cross-section at (1), it will be \( \int_{A_1} \rho_1 V_1 dA_1 \)

Similarly mass rate of flow through entire section (2) will be \( \int_{A_2} \rho_2 V_2 dA_2 \)

For steady flow, principle of conservation of mass gives:

\[
\int_{A_1} \rho_1 V_1 dA_1 = \int_{A_2} \rho_2 V_2 dA_2 = \text{Constant}
\]

For steady one dimensional flow where \( V_i \) and \( A_i \) represent the average velocity and cross-sectional area of section (1) and similarly \( v_2 \) and \( A_2 \) for section (2), then

\[
\rho_1 V_1 A_1 = \rho_2 V_2 A_2
\]

For incompressible flow \( \rho_1 = \rho_2 \)
Therefore,
\[ v_1 A_1 = v_2 A_2 = Q = \text{Constant} \quad (4.8) \]

Equation 4.8 is the continuity equation for steady, incompressible, one-dimensional flow.

\[ Q = A \cdot V = \left[ \frac{L^3}{T} \right] = \text{volume rate of flow} = \text{discharge}. \]

units of \( Q \): \( m^3/s \), \( l/sec \), \( ft^3/sec \), etc.

Example 4.2

A conical pipe has a diameter of 10 cm and 15 cm at the two ends respectively. If the velocity at the 10 cm end is 2 m/sec, what is the velocity at the other end and what is the discharge through the pipe?

Solution:
Continuity Equation:
\[ Q = A_1 v_1 = A_2 v_2 \]

given: \( v_1 = 2 \text{m/sec}, \ A_1 = \frac{\pi}{4} (0.1)^2, \ A_2 = \frac{\pi}{4} (0.15)^2 \)

\[ v_2 = v_1 \frac{A_1}{A_2} = 2 \times \frac{0.1^2}{0.15^2} = 0.89 \text{ m/sec} \]

discharge, \( Q = v_1 A_1 = 2 \times \frac{\pi}{4} (0.1)^2 = 0.0157 \text{ m}^3/\text{sec} \)

\[ = 15.7 \text{l/sec} \]
4.8.2 Two and Three Dimensional Flows

For the most general three dimensional case the continuity equation may be derived by considering an elemental volume of sides $\Delta x$, $\Delta y$ and $\Delta z$ in the cartesian co-ordinate system as shown in Fig. 4.10. Let the density at the centroid if the elemental space be $\rho$ and the components of velocity be $u$, $v$ and $w$ in the $x$, $y$, and $z$ directions respectively.

Consider first the mass inflow and outflow through the face normal to the $x$-axis:

Total mass inflow through the face on the left:

$$[\rho u - \frac{\partial (\rho u)}{\partial x} \frac{\Delta x}{2}] \Delta y \Delta z$$

Mass outflow through the opposite face:

$$[\rho u + \frac{\partial (\rho u)}{\partial x} \frac{\Delta x}{2}] \Delta y \Delta z$$
Hence the net rate of mass influx into the element through these faces is: \(-\frac{\partial (\rho u)}{\partial x} \Delta x \Delta y \Delta z\)

Similarly, net rate of mass influx through faces perpendicular to y axis is
\[-\frac{\partial (\rho v)}{\partial y} \Delta x \Delta y \Delta z\]

net rate of mass influx through faces perpendicular to z axis
\[-\frac{\partial (\rho w)}{\partial z} \Delta x \Delta y \Delta z\]

Total excess of mass passing into the element per unit time is:

\[-\left(\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z}\right) \Delta x \Delta y \Delta z\]

The rate of change of mass contained in the element is given by:

\[+\frac{\partial [\rho \Delta x \Delta y \Delta z]}{\partial t}\]

According to the principle of conservation of mass, the total excess rate of mass passing into the element should be equal to the rate of change of mass in the elemental volume:

\[\text{i.e. } -\left(\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z}\right) \Delta x \Delta y \Delta z = \frac{\partial (\rho \Delta x \Delta y \Delta z)}{\partial t}\]
If the elemental volume is allowed to shrink, i.e. \( \Delta x \Delta y \Delta z \to 0 \), then the general equation of continuity in Cartesian coordinates becomes:

\[
\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = -\frac{\partial p}{\partial t}
\]

For steady flow, it becomes:

\[
\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0
\]

For steady, incompressible flow; it will be:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (4.10)
\]

For two dimensional, steady, incompressible flow one gets:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (4.11)
\]

**Example 4.3**

Determine the value of \( v \) in a two-dimensional flow field when \( u = ax \).

**Solution:**

A possible flow should satisfy continuity equation,

\[
Since \quad u = ax, \quad \frac{\partial u}{\partial x} = a
\]
Two dimensional continuity equation is \[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

Therefore, \[ \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -a \]
\[ \partial v = -a \partial y \]

Integrating, \[ v = -\int a \partial y = -ay + f_1(x) \]
\[ \therefore \quad v = -ay + f_1(x) \]

Example 4.4

Does the velocity field given by \[ \vec{U} = 5x^3 \hat{i} - 15x^2 \hat{j} + tk \]
represent a possible fluid motion?

Solution:

Here, \[ u = 5x^3, \quad v = -15x^2y, \quad w = t \]

In order to check for a physically possible fluid motion, one needs to look for compliance with the Continuity equation.

For three-dimensional incompressible fluid, the continuity equation is:
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]
From the above, \[
\frac{\partial u}{\partial x} = 15x^2
\]
\[
\frac{\partial v}{\partial y} = -15x^2
\]
\[
\frac{\partial w}{\partial z} = 0
\]

Substitution in the continuity equation gives:

\[15x^2 - 15x^2 + 0 = 0\]

Since continuity equation is satisfied, the given velocity field represents a possible fluid motion.

4.9 Rotational and Irrotational Flows

The angular velocity of the fluid elements about their mass centres should be considered while discussing the kinematics of fluid flow. Accordingly, fluid motion in which the fluid particles do not rotate about their own axes is known as irrotational flow while fluid motion in which the fluid particles rotate about their own axes is known as rotational flow. These two types of flow are illustrated in figure 4.11

Figure 4.11(a) refers to an ideal (non-viscous) fluid flow between two parallel plates. The velocity distribution is uniform. If a stick abc is laid normal to a stream line 0-0, it does not undergo rotation about an axis normal to the plane of the paper through b and hence it does not change its orientation as it moves in the flow direction. Such a flow is termed as irrotational.
Figure 4.11 Irrotational and Rotational flows

Figure 4.11(b) shows a two dimensional real fluid flow with non-uniform velocity distribution with the velocities near the boundary being smaller than in the region close to the centre. A stick abc kept normal to 0-0 initially rotates about axis at b. Since the velocity at c is higher than at a, the stick will attain an inclined position after moving through a short distance. The stick has started rotating about an axis at b and its orientation has changed. Such a flow is termed as rotational.

Rotational and irrotational flows can be identified by determining the rotation of the fluid element at every point in a flow field.

Rotation: consider two elementary lengths such as OA and OB of length $\delta x$ and $\delta y$ respectively in a fluid as shown in Figure 4.12.

Let $v =$ velocity at o in the y direction.

$u =$ velocity at o in the x direction.

Thus, velocity at A in the y direction will be \[ v + \frac{\partial v}{\partial x} \delta x \]

velocity at B in the x direction will be \[ u + \frac{\partial u}{\partial y} \delta y \]
Since velocities at O and A are different in the y direction, OA will rotate in the counter clockwise (+ive) direction.

Thus, the angular velocity of OA  
\[
\omega = \frac{(v + \frac{\partial v}{\partial x} \delta x) - v}{\delta x} = \frac{\partial v}{\partial x}
\]

Similarly OB will rotate in the clockwise (-ive) direction.

The angular velocity of OB  
\[
\omega = \frac{u + \frac{\partial u}{\partial y} \delta y - u}{\delta y} = -\frac{\partial u}{\partial y}
\]

If the rotation about the z-axis (i.e in the x-y plane), \( \omega_z \), is defined as the average rotation of the two elements OA and OB, then:

\[
\omega_z = \frac{1}{2} \left[ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right] \quad (4.12)
\]
Similarly:

$$\omega_x = \frac{1}{2} \left[ \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right]$$  \hspace{1cm} (4.13)

$$\omega_y = \frac{1}{2} \left[ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right]$$  \hspace{1cm} (4.14)

If at every point in a flow field the rotations $\omega_x$, $\omega_y$ and $\omega_z$ are zero, then the flow is known as irrotational; otherwise, the flow is rotational.

Thus for a two dimensional flow in the x-y plan, $\omega_z = 0$ leading to $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$ as the condition of irrotationality.

Example 4.5

The velocity components of a two dimensional flow in the x-y plane are: $u = -3y^2$ and $v = -4x$. Does this represent a possible flow? If the flow is possible is it rotational or irrotational?

Solution:

Continuity equation in two-dimensional incompressible flow is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Here, $\frac{\partial u}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 0$
Therefore, continuity is satisfied and the flow is possible. To check whether the flow is rotational or irrotational, use the equation for rotation about the z-axis which is:

\[ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0, \]

then the flow is irrotational otherwise, it is rotational.

Thus
\[
\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -4 - (-6y) = -4 + 6y \neq 0
\]

Therefore, the flow is rotational.

4.10 Stream Function

Consider the streamline pattern of a two dimensional, steady, incompressible flow shown in Figure 4.13.

Let A be a fixed position and B a variable position in the flow field. A and B may be joined by arbitrary curves such as APB and AOB. Let the thickness of the flow field in the Z direction be unity.
Then, the rate of flow through curve APB is equal to the rate of flow through AQB since the flow between two streamlines must remain unchanged. But the rate of flow depends upon the positions of A and B. If A is fixed, the rate of flow becomes a function of the position of B only. This function is known as stream function and denoted by $\Psi$. If the value of $\Psi$ at A is zero, then the value of $\Psi$ at B represents the flow rate between positions A and B. Consider any other point B' along the streamline through B. Since no flow occurs across BB', the flow through AB' should be the same as the flow through APB. Hence, the value of $\Psi$ at B' should be the same as at B. Thus, the value of the stream function $\Psi$ is constant along a streamline. Each streamline will have a different value of $\Psi$ such that the difference in $\Psi$ values of two streamlines gives the flow rate between the two streamlines.

The relationship between the velocity components in the x and y directions of a two dimensional flow and the stream function $\Psi$ may be developed by considering the two streamlines shown in Figure 4.14.

![Figure 4.14](image)

Let the value of the stream function for streamline AB be $\Psi$ and the value of the stream function for streamline CD be $\Psi + \Delta\Psi$. The normal distance between the two stream lines is $\Delta n$. Then,
\( \Delta \psi = q \Delta n \), where \( q \) = average velocity of flow at section nm.

As \( \Delta n \to 0 \), \[ q = \frac{\partial \psi}{\partial n} \]

Note that what is required is the difference in the value of the stream functions between two streamlines, and not the absolute value of \( \psi \), in the determination of the velocity of the discharge. Hence the value \( \psi = 0 \) can be assigned to any streamline. Flow rate across length mn equals the sum of the flow rates across mp and np.

Since \( \psi = \psi(x,y) \),

\[ d\psi = \frac{\partial \psi}{\partial x} \delta x + \frac{\partial \psi}{\partial y} \delta y \]

Taking the velocity of flow across np (= \( \delta y \)) to be \( u \) and that across mp (= \( \delta x \)) to be \( -v \), it is clear that:

\[ d\psi = q \Delta n = u \delta y - v \delta x \]

Comparing this equation with the above total differential of \( \psi \), it is clear that:

\[ u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \]  \hspace{1cm} (4.15)

Equation 4.15 is the relationship between the stream function in the x-y plane and the velocity components in the x and y directions. Examination of the continuity equation for two-dimensional incompressible flow i.e
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

in light of the relations expressed in equation 4.15 and substitution gives:

\[ \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = 0 \]

which shows that the continuity equation is identically satisfied. Therefore, the existence of a stream function \( \psi \) for a flow implies a possible flow and conversely, for any possible flow, a stream function \( \psi \) must exist. Considering the equation of a streamline in the \( x-y \) plane.

\[ i.e. \quad \frac{dx}{u} = \frac{dy}{v} \]

or \[ udy - vdx = 0 \]

Substituting the values of \( u \) and \( \psi \) from equation 4.15,

\[ \frac{\partial \psi}{\partial y} \cdot dy + \frac{\partial \psi}{\partial x} \cdot dx = 0 \]

The left hand side of the above equation is the total differential \( d\psi \) of \( \psi = f(x,y) \).

Since \( d\psi = 0 \), \( \psi = c = \text{constant} \) along a streamline.
Example 4.6

A stream function is given by $\psi = x + y^2$. Determine the magnitude of the velocity components in the x and y directions at (1,3).

Solution:

\[ u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y} (x + y^2) = 2y \]
\[ v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x} (x + y^2) = -1 \]

The above stream function represents a possible flow since the continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 + 0 = 0$.

At point (1,3)

\[ u = 2 \times 3 = 6 \]
\[ v = -1 \]

4.11 Velocity Potential

Analogous to the principle that electric current flows in the direction of decreasing voltage and that the rate of flow of current is proportional to the difference in voltage potential between two points, the velocity of flow of a fluid in a particular direction would depend on certain potential difference called velocity potential. The velocity potential, denoted by $\phi(\phi)$, decreases in the direction of flow. It has no absolute value and is simply a scalar function of position and time. For steady flow, the velocity components u,v and w
in the x, y and z directions respectively, in terms of velocity potential are:

\[ u = \frac{-\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y} \quad \text{and} \quad w = \frac{-\partial \phi}{\partial z} \]  \hspace{1cm} (4.16)

A potential line is a line along which the potential \( \phi \) is constant. Thus if a potential function exists for a certain flow, then it is possible to draw lines of constant potential.

Some of the properties of the potential function \( \phi \) may be derived by substituting it in the equations of rotation and continuity. Substituting the values of the velocity components \( u, v \) and \( w \) of equation 4.16 in the expression for rotation, one obtains:

\[
\omega_x = \frac{1}{2} \left[ \frac{\partial w}{\partial z} - \frac{\partial v}{\partial z} \right] = \frac{1}{2} \left[ \frac{\partial^2 \phi}{\partial z \partial y} - \frac{\partial^2 \phi}{\partial y \partial z} \right]
\]

\[
\omega_y = \frac{1}{2} \left[ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right] = \frac{1}{2} \left[ \frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x} \right]
\]

\[
\omega_z = \frac{1}{2} \left[ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right] = \frac{1}{2} \left[ \frac{\partial^2 \phi}{\partial y \partial x} - \frac{\partial^2 \phi}{\partial x \partial y} \right]
\]

If \( \phi \) is a continuous function, \( \frac{\partial^2 \phi}{\partial z \partial y} = \frac{\partial^2 \phi}{\partial y \partial z} \) etc.

Hence \( \omega_x = \omega_y = \omega_z = 0 \), which is the condition for irrotationality. Therefore, if a velocity potential \( \phi \) exists then the flow should be irrotational and vice versa.
Substitution of the velocity components given in equation 4.16 in the three dimensional continuity equation 4.10 leads to the Laplace Equation:

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \] (4.17)

Hence, any function \( \phi \) which satisfies the Laplace Equation is a case of steady, incompressible, irrotational flow and such a flow is known as potential flow.

It should be noted that the stream function \( \Psi \) applies both for rotational and irrotational flows. However the potential function \( \phi \) is applicable only for irrotational flow. Equations 4.15 and 4.16 may be used to establish the relationship between stream function \( \Psi \) and potential function \( \phi \) for an irrotational, steady, incompressible flow leading to the following:

\[ \frac{\partial \psi}{\partial y} = -\frac{\partial \phi}{\partial x} \]

\[ \frac{\partial \psi}{\partial x} = +\frac{\partial \phi}{\partial y} \] (4.18)

Equations 4.18 are known as the Cauchy-Riemann Equations.

Example 4.7

Show that \( \psi = x^2 - y^2 \) represents a case of two dimensional flow and find its potential function.
Solution:

From \( \psi = x^2 - y^2 \),

\[
    u = \frac{\partial \psi}{\partial y} = -2y
\]

\[
    v = -\frac{\partial \psi}{\partial x} = -2x
\]

The two dimensional continuity equation will be:

\[
    \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 + 0 = 0
\]

Thus continuity is satisfied and the stream function represents a case of two dimensional flow.

Further:

\[
    \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial x} \right) = \frac{\partial}{\partial x} (2x) = 2
\]

\[
    \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial y} \right) = \frac{\partial}{\partial y} (-2y) = -2
\]

Therefore, \( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 2 - 2 = 0 \)

Thus, since \( \psi \) satisfies the Laplace equation the flow is also irrotational.
The velocity potential that satisfies both (a) and (b) is:
\[ \phi = 2xy + C, \] where \( C \) is constant.

Example 4.8

In a two dimensional, incompressible flow velocity components are given by: \( u = x - 4y \) and \( v = -y - 4x \). Show that the flow satisfies the continuity equation and obtain the expression for the stream function. If the flow is potential obtain also the expression for the velocity potential.

Solution:

For an incompressible, two dimensional flow, the continuity equation is:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

Here, \( u = x - 4y \) and \( v = -(y + 4x) \)

Therefore, \( \frac{\partial u}{\partial x} = 1, \) and \( \frac{\partial v}{\partial y} = -1 \)

Thus, \[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 1 + (-1) = 0 \]
i.e. the flow satisfies continuity equation.

To obtain the stream function,

\[ u = \frac{\partial \psi}{\partial y} = x - 4y \]  \quad (i)

\[ v = -\frac{\partial \psi}{\partial x} = +(y + 4x) \]  \quad (ii)

from (i):
\[ \psi = \int (x - 4y) \, dy = xy - 2y^2 + f(x) + C \]

But if \( \psi_o = 0 \) at \( x = 0 \) and \( y = 0 \), then the reference streamline passes through the origin, then \( C = 0 \)

Then \( \psi = xy - 2y^2 + f(x) \) \quad (iii)

Differentiating (iii) with respect to \( x \) and equation to \(-v\),
\[ \frac{\partial \psi}{\partial x} = y + \frac{\partial}{\partial x} (f(x)) = y + 4x \]
\[ \therefore f(x) = \int 4x \, dx = 2x^2 \]

Thus, \( \psi = 2x^2 + xy - 2y^2 \)

To check whether the flow is potential, the Laplace equation must be satisfied i.e.
\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \]

from \( \psi = 2x^2 + xy - 2y^2 \),

\[ \frac{\partial^2 \psi}{\partial x^2} = 4, \quad \text{and} \quad \frac{\partial^2 \psi}{\partial y^2} = -4 \]

Thus \( 4 - 4 = 0 \) showing that the flow is potential.

To obtain the velocity potential,

\[ \frac{\partial \phi}{\partial x} = -u = -(x - 4y) = 4y - x \]

Therefore, \( \phi = \int (4y - x) \, dx = 4yx - \frac{x^2}{2} + f(y) + C \)

But \( f_c = 0 \) at \( x = 0 \) and \( y = 0 \), so that \( C = 0 \)

Thus \( \phi = 4yx - \frac{x^2}{2} + f(y) \)

\[ \frac{\partial \phi}{\partial y} = 4x + \frac{d}{dy} (f(y)) = -v = 4x + y \]

Therefore, \( \frac{d}{dy} (f(y)) = y \)

or \( f(y) = \frac{y^2}{2} \)

Thus \( \phi = \frac{y^2}{2} + 4yx - \frac{x^2}{2} \)
4.12 Flow net

For any two-dimensional irrotational flow of an ideal fluid, two series of lines may be drawn as shown in Figure 4.15. These are: streamlines i.e. lines along which $\psi$ is constant and equipotential lines i.e. lines along which $\phi$ is constant.

Consider the equipotential line AB along which $\phi$ is constant and equal to say 3. Since $\phi$ is constant along such a line, the velocity tangential to such a line, $-\frac{\partial \phi}{\partial n} = v_n = 0$. However, since $\phi$ is varying in the s direction, the velocity normal to the equipotential line, $\frac{\partial \phi}{\partial s} = v_s$ exists. Equipotential lines thus have the property that the flow is always at right angle to them.

Along streamlines such as CD, the stream function $\psi$ is constant. Therefore, $\frac{\partial \psi}{\partial s} = v_n = 0$ i.e. there is no flow at
right angles to the $\Psi = \text{constant}$ line. But $-\frac{\partial \phi}{\partial n} = v_s$ exists.

Hence, the flow is always tangential to the $\Psi = \text{constant}$ line. Thus, at every point along a streamline, the velocity is always tangential to it. This shows that tangents to a streamline and an equipotential line intersect at $90^\circ$. A series of streamlines and equipotential lines form an orthogonal grid. Such a system represented graphically by finite number of equipotential and stream lines is called a flow net. The flow net is composed of a family of equipotential lines and a corresponding family of streamlines with the constants varying in arithmetic progression.

It is customary to let the change in constant between adjacent equipotential lines and adjacent streamlines be the same. If, at some small region of the flow net, the distance between adjacent streamlines $= \Delta n$ and that between adjacent equipotential lines $= \Delta s$, then the approximate velocity $v_s$ (in the s direction) is given by:

$$v_s = -\frac{\Delta \phi}{\Delta s},$$

in terms of the spacing of equipotential lines

and

$$v_s = \frac{\Delta \psi}{\Delta n},$$

in terms of the spacing of streamlines.

Thus: $\Delta s = \Delta n$, since $\Delta \phi = -\Delta \psi$.

The flow net thus consists of an orthogonal grid that reduces to perfect squares in the limit as the grid size approaches zero. For a given set of boundary conditions there is only one possible pattern of flow of an ideal fluid.
Example 4.9

The potential function of a two dimensional irrotational flow is given by \( \phi = Ax \) where \( A \) is a constant. Determine the stream function \( \Psi \) and draw a set of streamlines and equipotential lines.

Solution:

First check if \( \phi = Ax \) satisfies the Laplace Equation.

\[
\frac{\partial \phi}{\partial x} = A \quad \text{and} \quad \frac{\partial^2 \phi}{\partial x^2} = 0 ; \quad \frac{\partial \phi}{\partial y} = 0 \quad \text{and} \quad \frac{\partial^2 \phi}{\partial y^2} = 0
\]

Therefore,

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0
\]

Hence, \( \phi = Ax \) represents a fluid flow case.

Next, find the stream function \( \Psi \)

Since \( -\frac{\partial \phi}{\partial x} = u = -A \) and \( u = -\frac{\partial \Psi}{\partial y} \),

then \( \frac{\partial \Psi}{\partial y} = A \)

integrating, \( \Psi = Ay + f(x) \)

Differentiating the above expression for \( \Psi \) with respect to \( x \),

\[
\frac{\partial \Psi}{\partial x} = f'(x). \quad \text{But} \quad \frac{\partial \Psi}{\partial x} = v \quad \text{and} \quad -\frac{\partial \phi}{\partial y} = v = 0
\]

Therefore, \( f'(x) = v = 0 \)

\[
\text{Hence, } f(x) = \text{constant}
\]

Thus \( \Psi = Ay + C \), where \( C \) is a constant
The corresponding streamlines and equipotential lines are shown in Figure 4.16 representing the case of parallel flow where the streamlines are parallel to the x-axis.

\[ \phi = \text{Constant} \]
\[ \psi = \text{Constant} \]

\[ y \]
\[ x \]

Figure 4.16

4.12.1 Construction of Flow Nets

It is clear from the foregoing discussions that obtaining the flow pattern or flow net for steady two dimensional irrotational flow involves the solution of the Laplace equation with given boundary conditions. The Laplace equation is a second order linear partial differential equation. Thus for flow patterns that can be interpreted as results of combinations of simple patterns, the superposition of the solutions of the Laplace equation for the simple patterns can lead to the determination of the given flow pattern.

However, there are complex flow patterns of practical interest that are too involved for analytical solutions. In such cases fairly good approximations may be obtained by using mostly graphical method and electrical analogy method. The graphical method of construction of flow nets is presented and discussed below.
Graphical Method:

The graphical method is illustrated by considering the flow through the transition shown in Figure 4.17.

Since there is no flow at right angles to the boundary, the fixed boundaries AB and CD coincide with streamlines. Far to the left of section AC and to the right of section BD, the flow is uniform and therefore the streamlines will be equally spaced in these regions. Decide first on the number of streamlines. The more the number of streamlines, the more accurate will be the result, but more time will be spent in constructing the net. AC and BD represent equipotential lines. Mark equally spaced points on AC and BD representing the intersection of the streamlines with the equipotential lines through these sections. Between these corresponding points, streamlines can be joined by smooth curves. In the narrower section BD, the spacing between the streamlines is narrower than the wider section AC and from continuity, the velocities at section BD will be higher than at section AC. Equipotential lines can now be drawn such that:

- they intersect the boundary and other streamlines at right angles.
- the distance between consecutive streamlines and the
distance between consecutive equipotential lines are equal so that the two form squares.
- the diagonals of the squares form smooth curves which intersect each other normally.

At the curved boundary, the latter two conditions may not be completely satisfied unless the spacings are very close or the mesh is very fine. Successive trials may be required to arrive at a satisfactory flownet. The flownet so drawn will be the same for various discharges and for geometrically similar transitions of various sizes. Typical flow nets are shown in Figure 4.18.
4.12.2 Uses of the Flow Net

A flow net of a two dimensional flow field under a given boundary condition, drawn to represent the flow pattern using a finite set of stream and equipotential lines will have the following uses:

i) The velocity at any point in the flow field can be determined if the velocity at a given point is known using continuity of flow between two streamlines.

ii) The flow net enables the determination of the velocity distribution and the pressure distribution, the knowledge of which is necessary to calculate drag forces and uplift forces.

iii) It makes the visualization of flow pattern possible thus enabling the modification of boundaries to avoid undesirable effects such as separation and stagnation.
EXERCISE PROBLEMS

4.1 A 200 mm diameter pipe bifurcates into a 120 mm diameter pipe and a 100 mm diameter pipe. If the flow through the 200 mm diameter pipe is 100 l/s assuming the velocity in the branch pipes are equal, find the rate of flow through each of the branch pipes. (Ans. 59.1 l/s; 40.1 l/s)

4.2 A diffuser at the end of a 100 mm diameter pipe is as show in Figure P 4.2. If the rate of flow through the pipe is 0.1 m³/s, find the exit velocity at the diffuser. What is the ratio between the exit velocity at diffuser and the velocity in the 100 mm diameter pipe?

4.3 In a two dimensional incompressible flow the x-component of the velocity is given by \( u = 3x-y \). Using the continuity equation, find the velocity component in the y-direction. (Ans. \( v = -3y \))

4.4 The velocity components in a two-dimensional flow are expressed as:
\[
\begin{align*}
u &= y^3/3 + 4x - x^2y; \\
v &= xy^2 - 4y - x^3/3
\end{align*}
\]
show that these functions represent a possible case of irrotational flow.
4.5 For a three dimensional flow, \( u = x^2 + z^2 + 5 \) and \( v = y^2 + z^2 \). Determine \( w \). (Ans. \( w = -2(x + y)z \))

4.6 Determine the stream function for a fluid flow if \( u = 2x \) and \( v = -2y \). Determine also the potential function.

4.7 If for a two dimensional potential flow, the velocity potential is given by \( \phi = x(2y - 1) \)

i) Determine the velocity at point \( p(4,5) \)

ii) What is the value of the stream function \( \Psi \) at point \( p \)?

(Ans. i) \( u = 9 \) and \( v = 8 \) units

(ii) \( \Psi = y^2 - x^2 - y \), 4 units)

4.8 State if the flow represented by: \( u = 3x + 4y \) and \( v = 2x - 3y \) is rotational or irrotational. Find the potential function if the flow is irrotational and the vorticity if it is rotational.
CHAPTER 5
DYNAMICS OF FLUID FLOW

5.1 Introduction

Dynamics of fluid flow deals with the forces responsible for fluid motion, the resulting accelerations and the energy change involved in the flow phenomenon.

Just as in mechanics of solids, the mechanics of fluids is also governed by Newton's Second Law of motion i.e

\[
\text{Force} = \text{Mass} \times \text{acceleration}
\]

The force and the acceleration are in the same direction. However, since liquids do not possess regidity of form, their mass center changes unlike that of solids. Therefore, for fluids mass per unit volume is more important than the total mass.

Thus in the x direction, Newton's Equation of motion will be:

\[
\sum F_x = \rho a_x
\]

where: \( \Sigma F_x \) = sum of x components of all forces per unit volume acting on the fluid mass.

\( a_x \) = total acceleration in the x direction.

\( \rho \) = mass per unit volume of the fluid.

5.2 Forces Influencing Motion

In general the following different types of forces influence fluid motion: Force due to gravity, Pressure,
viscosity, Surface tension, Compressibility, and Turbulence.

**Gravity force** $F_g$: is due to the weight of the fluid. Its component in the direction of motion causes acceleration in problems where gravity is important such as in open channel flow. $F_g$ is proportional to the volume of the fluid mass under consideration. The gravity force per unit volume, $f_g = \rho g$ and acts vertically downwards.

**Fluid pressure force** $F_p$: This is the force exerted by a fluid mass on any surface in a direction normal to the surface. The pressure intensity $p$ is the force per unit area and indicates a local intensity of pressure force. Fluid pressure produces acceleration in a given direction only if the pressure decreases in that direction.

To determine the magnitude of the pressure force per unit volume, consider a small fluid element of cross-sectional area $dA$ and length $dx$ as shown in Fig 5.1.

![Figure 5.1 Pressure forces](image)

$P$ is the pressure on the left face

$\frac{\partial p}{\partial x}$ is rate of change of pressure in the $x$ direction
\[ P + \frac{\partial p}{\partial x} \, dx \] is pressure on the right face

Since there is a difference in pressure between the two faces, there exists a pressure force \( F_p \) in the x direction which can cause the fluid to move in the x direction;

\[ F_{px} = p\,dA - (p + \frac{\partial p}{\partial x} \, dx) \, dA = -\frac{\partial p}{\partial x} \, dx \cdot dA \]

Thus the pressure force per unit volume is \( F_{px} = -\frac{\partial p}{\partial x} \)

The negative sign indicates that \( F_{px} \) acts in the direction of decreasing pressure.

**Viscous force \( F_v \):** This force exists in all real fluids. When there is relative motion between two layers of a fluid, a tangential force is created due to the effects of viscosity. The shear resistance, \( F_v \), acts in a direction opposite to that of motion thus retarding the flow.

**Surface tension force \( F_s \):** This force is important when the depths of flow or the related length dimensions are extremely small. \( F_s = \text{surface tension force/volume} \).

**Force due to compressibility \( F_c \):** For incompressible fluids this becomes significant in problems of unsteady flow like water hammer where the elastic properties of fluids come into the picture.

In most problems, \( F_v \) and \( F_c \) are neglected

**Forces due to turbulence \( F_t \):** In highly turbulent flows, there is a continuous momentum transfer between adjacent layers which
causes normal and shear stresses due to turbulence. These are known as Reynolds stresses. These stresses, designated by F, must be taken into consideration in cases of turbulent flow.

5.3 Euler's Equation of Motion

Fluid motion is influenced by all the forces mentioned above. Motion of a fluid in any direction is thus caused by the components of all the forces in that direction. Thus for the x direction, Newton's Second Law will give the following:

\[ F_{gx} + F_{px} + F_{vx} + F_{ox} + F_{ex} + F_{tx} = \rho a_x \]  \hspace{1cm} (5.1)

Similar equations can be written for the other two coordinate directions.

For ideal fluids which have no viscosity and neglecting F_{ox}, F_{ex} and F_{ux}, the equation 5.1 reduces to:

\[ F_{gx} + F_{px} = \rho a_x \]  \hspace{1cm} (5.2)

When proper expression for F_{gx}, F_{px} and a_{x} are substituted in Eqn. 5.2 it becomes what is known as Euler's Equation of Motion in the x direction.

When viscous forces are considered, Equation 5.2 will be:

\[ F_{gx} + F_{px} + F_{vx} = \rho a_x \]  \hspace{1cm} (5.3)

This equation gives the Navier-Stoke's Equation in the x direction when the proper expressions for the forces and accelerations are substituted in it. For turbulent flow, the force due to turbulence must also be considered in addition to
the forces in Equation 5.3. Thus, we have

$$F_{gx} + F_{px} + F_{vx} + F_{tx} = \rho a_x$$  \hspace{1cm} (5.4)

This gives the Reynold's Equation of motion in the x direction.

In this discussion, we will consider Euler's Equation in some detail.

In Equation 5.2, introducing $-\frac{\partial p}{\partial x}$ to replace $F_{px}$ and letting $F_{gx}$ represent the gravity force per unit volume, we will have:

$$F_{gx} - \frac{\partial p}{\partial x} = \rho \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right]$$  \hspace{1cm} (5.5)

Equation 5.5 is Euler's Equation for one dimensional flow.

The general three dimensional form of Euler's Equation of Motion can be written as:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = X - \frac{1}{\rho} \frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = Y - \frac{1}{\rho} \frac{\partial p}{\partial y}$$  \hspace{1cm} (5.6)

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = Z - \frac{1}{\rho} \frac{\partial p}{\partial z}$$

$u, v$ and $w$ are velocity components in the $x, y$ and $z$ directions and $X, Y$ and $Z$ are components of fluid weight per unit mass in the $x, y$ and $z$ directions respectively.
5.4 Integration of Euler's Equation of Motion

The integration of Euler's Equation along a streamline results in an important equation in fluid mechanics known as Bernoulli's Equation.

Consider the forces acting on a fluid element of cross-sectional area dA and length ds along a streamline as shown in figure 5.2.

![Figure 5.2 Forces acting on a fluid element](image)

The forces acting on the fluid element are those due to gravity and due to pressure gradient. It is assumed that the fluid is frictionless and all minor forces are neglected. Thus:

Gravity force in the direction of motion = $\rho g dA ds \cos \theta$

Pressure force in the direction of motion = $-\frac{\partial p}{\partial s} ds dA$

If $v$ = velocity in the direction of motion, then Euler's Equation 5.5 becomes:

$$\rho g dA ds \cos \theta - \frac{\partial p}{\partial s} ds dA = \rho dA ds \left( \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial s} \right) \quad (5.7)$$
Dividing Equation 5.7 by $\rho dA ds$, one gets:

$$g \cos \theta - \frac{1}{\rho} \frac{\partial p}{\partial s} = \left( \frac{\partial v}{\partial t} = v \frac{\partial v}{\partial s} \right)$$

Considering for steady flow, $\frac{\partial v}{\partial t} = 0$ and substituting $\cos \theta = -\frac{\partial z}{\partial s}$, the above equation becomes:

$$-g \frac{\partial z}{\partial s} - \frac{1}{\rho} \frac{\partial p}{\partial s} = v \frac{\partial v}{\partial s} \quad (5.8)$$

Equation 5.8 can also be written as:

$$\rho v \frac{\partial v}{\partial s} = -\frac{\partial}{\partial s} (P + \gamma z) \quad (5.8a)$$

Equation 5.8 can be integrated along a streamline after multiplying each term by $ds$. Hence:

$$-\int g dz - \int \frac{dp}{\rho} = \int v dv \quad (5.9)$$

Thus: $\frac{v^2}{2} + gz + \int \frac{dp}{\ell} = \text{Constant}$

Equation 5.9 is Bernoulli's Equation for both compressible and incompressible fluids.

For incompressible fluid, $q$ is independent of pressure i.e. $q$ is constant.
Hence, \[ \frac{v^2}{2} + gz + \frac{P}{\gamma} = \text{Constant} \] \hspace{1cm} (5.10)

Equation 5.10 is Bernoulli's Equation for incompressible fluids. It can be written in the following two alternative forms:

\[ \frac{v^2}{2g} + z + \frac{P}{\gamma} = \text{Constant} \] \hspace{1cm} (5.11)

or \[ \frac{\rho v^2}{2} + \gamma z + p = \text{Constant} \] \hspace{1cm} (5.12)

Each term in Equation 5.10, 5.11 and 5.12 represents energy of the fluid.

The terms in Equation 5.10 describe the energy per unit mass. The terms in Equation 5.11 describe the energy per unit wt. The terms in Equation 5.12 describe the energy per unit volume.

Mostly, however, Equation 5.11 is used in pipe and open channel flows. Each terms in Equation 5.11 has the dimension of length.

\[ \frac{v^2}{2g} \] is known as the velocity head (Kinetic Energy per unit weight),

\[ \frac{P}{\gamma} \] is known as the pressure head (pressure energy per unit weight),

and

\[ z \] is known as the elevation or potential head (potential head per unit weight).
Bernoulli's Equation states that in a steady flow of an ideal fluid, the sum of velocity head, pressure head and potential head along a stream line is constant. Applying it between two sections,

\[
\frac{v_1^2}{2g} + \left(\frac{p_1}{\gamma} + z_1\right) = \frac{v_2^2}{2g} + \left(\frac{p_2}{\gamma} + z_2\right)
\]

or Bernoulli's Equation can also be stated as: The total energy per unit weight for a steady flow of an ideal fluid remains constant along a stream line.

5.5 The Energy Equation

For real fluids, some energy is converted into heat due to viscous shear and consequently there is a certain amount of energy loss. Thus, for real fluids, equation 5.13 becomes:

\[
\frac{v_1^2}{2g} + \frac{p_1}{\gamma} + z_1 = \frac{v_2^2}{2g} + \frac{p_2}{\gamma} + z_2 + h_{\text{loss}}
\]

Total energy at (1) = Total energy at (2) + Loss of energy between (1) and (2)

where subscripts (1) and (2) refer to the two sections under consideration and \( h_{\text{loss}} \) is the energy loss per unit weight of fluid between the two sections. Section (1) is the upstream and section (2) is the downstream section and flow takes place from section (1) towards section (2). Equation 5.14 is the energy equation for real fluid flow.

If energy is added to the fluid between sections (1) and (2) such as by a pump, then the energy equation will be:
Where $H_p$ is the energy head added by the pump.

If energy is taken out of the system between sections (1) and (2) by a turbine, the energy equation will be:

\[
\frac{V_1^2}{2g} + \frac{P_1}{\gamma} + z_1 + H_t = \frac{V_2^2}{2g} + \frac{P_2}{\gamma} + z_2 + h_{L_1-2} \quad (5.16)
\]

Where $H_t$ is the head supplied to the turbine.

In all these cases, the velocity is assumed to be constant throughout the cross-section. This assumption may be valid in turbulent flows where the average velocity is not very different from the maximum. With varying velocities across a cross-section, a kinetic energy correction factor $\alpha$ should be applied to the kinetic energy head term. Referring to the velocity distribution at a cross-section shown in Figure 5.3:

![Figure 5.3](image)

Let $v$ = the velocity at a particular point in a cross-section where the elemental flow area is $dA$. 

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Then, the kinetic energy per unit weight = \( \frac{v^2}{2g} \)

Weight of fluid passing through \( dA \) per unit time = \( \gamma . v . dA \)

Kinetic energy passing through \( dA \) per unit time = \( \gamma . v . dA \cdot \frac{v^2}{2g} \)

The integral of the above expression gives the total kinetic energy passing through the whole cross-sectional area \( A \) per unit time.

Using the mean velocity \( \bar{v} \) and the kinetic energy correction factor \( \alpha \), the kinetic energy per unit time passing through the section will be \( \gamma . \bar{v} . A \cdot \frac{\bar{v}^2}{2g} \).

Thus:

\[
\gamma . \bar{v} . A . \alpha \cdot \frac{\bar{v}^2}{2g} = \gamma \int_A \frac{v^2}{2g} . v . dA
\]

or

\[
\alpha = \frac{1}{A \bar{v}^3} \int_A v^3 . dA
\]

Once the kinetic energy correction factor for a particular velocity distribution is known, then Bernoulli's equation between two sections (1) and (2) becomes:

\[
z_1 + \frac{p_1}{\gamma} + \alpha_1 \frac{v_1^2}{2g} = z_2 + \frac{p_2}{\gamma} + \alpha_2 \frac{v_2^2}{2g}
\]
For laminar flow in a pipe, $\alpha = 2$. For turbulent flow in a pipe, $\alpha$ varies from about 1.01 to 1.10 and is usually taken as unity except for precise work.

Example 5.2

Calculate the kinetic energy correction factor $\alpha$ for a parabolic velocity distribution in a pipe flow of radius $r_0$ given by:

$$v = v_{\text{max}} \left(1 - \frac{r^2}{r_o^2}\right)$$

Solution:

Referring to the figure below:

![Figure E 5.2](image-url)
Mean Velocity \( \bar{v} = \frac{Q}{A} = \frac{\int_A v dA}{A} \)

\[
\begin{align*}
\int_0^{r_o} v_{\text{max}} \left(1 - \frac{r^2}{r_o^2}\right) \cdot 2\pi r dr &= \frac{2 \pi r_o^2 v_{\text{max}}}{\pi r_o^2} \\
&= \frac{2 v_{\text{max}}}{r_o^2} \left( r - \frac{r^3}{r_o^2} \right) dr \\
&= \frac{2 v_{\text{max}}}{r_o^2} \left( \frac{r^2}{2} - \frac{r^2}{4} \right) = \frac{v_{\text{max}}}{2}
\end{align*}
\]

\[\alpha = \frac{1}{A v^3} \int_A v^3 dA\]

\[
\int_A v^3 dA = \int_0^{r_o} v_{\text{max}}^3 \left(1 - \frac{r^2}{r_o^2}\right)^3 \cdot 2\pi r dr
\]

\[
\begin{align*}
&= 2\pi v_{\text{max}}^3 \int_0^{r_o} \left(r - \frac{3r^3}{r_o^2} + \frac{3r^5}{r_o^4} - \frac{r^7}{r_o^6}\right) dr \\
&= 2\pi v_{\text{max}}^3 r_o^2 \left( \frac{1}{2} - \frac{3}{4} + \frac{3}{6} - \frac{1}{8} \right) = \frac{\pi}{4} v_{\text{max}}^3 r_o^2
\end{align*}
\]

\[\therefore \alpha = \frac{\frac{\pi}{4} v_{\text{max}}^3 r_o^2}{\pi r_o^2 \left(\frac{v_{\text{max}}}{2}\right)^3} = 2\]
Example 5.3

A closed tank of a fire engine is partly filled with water, the air space above the water being under pressure. A 5 cm diameter hose connected to the tank discharges on the roof of a building 4.0 m above the level of water in the tank. The frictional head loss in the hose is equivalent to 40 cm head of water. What air pressure must be maintained in the air in the tank to deliver 12ℓ/s on the roof?

Solution:

Referring to Figure E 5.3

The discharge in the hose = 12ℓ/s = 0.012 m³/s

Velocity V in the 5 cm hose = \( \frac{\text{0.012}}{\frac{\pi}{4}(0.05)^2} \) = 6.1 m/s

\[ \therefore \text{The velocity head} \quad \frac{V^2}{2g} = \frac{6.1^2}{2 \times 9.81} = 1.9 \text{ m} \]

Applying the energy equation between (1) and (2) taking the water level in the tank as datum:

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Thus, a pressure of 61.80 kN/m² must be maintained in the air above the water in the tank to deliver 12 l/s of water on the roof.

5.6 Power Considerations

When water under pressure is lead through a turbine, hydraulic energy is converted to mechanical energy, in the form of turbine rotation, which may then be converted to electrical energy by means of a generator coupled to the turbine. Thus energy is extracted from the water. Converely, a pump adds mechanical energy to the water which enables it to be lifted from a lower level to a higher level reservoir or makes the transportation of the water from one location to another possible by overcoming resistance to flow in the piping system. The power extracted from or added to the water may be calculated from the following:
Power \( P = \text{Work done per unit time} \)

\[ = G.H \]

\[ = \gamma_w QH \quad (5.17) \]

Where \( G \) is the weight rate of flow in N/s
\( H \) is the energy head extracted or added in m
\( Q \) is the discharge in m\(^3\)/s
\( \gamma_w \) is specific weight of water in N/m\(^3\)

The power \( P \) is in Nm/s. Since 1 nm/s is equal to 1 watt, the power \( P \) in Equation 5.17 is in watts when \( \gamma_w \) is in N/m\(^3\), \( Q \) is in m\(^3\)/s and \( H \) is in meters. But in the MKS system \( \gamma_w \) is in kgf/m\(^3\) and \( \gamma_w = 1000 \) kgf/m\(^3\).

Since 1 metric Horse power = 75 kg-m/s, then

\[ P(\text{in Horse power}) = \frac{\gamma_w QH}{75}, \quad \text{where } \gamma_w = 1000 \text{ kgf/m}^3 \]

\( H \) is in m
\( Q \) is in m\(^3\)/s

If the efficiency of a turbine to convert hydraulic energy to mechanical energy in \( \eta_t \) and the head of water extracted from the flowing water by the turbine is \( H_t \) metres of water, then the power in Horse power supplied to the generator is:

\[ P = \frac{\gamma_w QH_t}{75} \cdot \eta_t \]

If the efficiency of a pump to convert mechanical energy to hydraulic energy of the water is \( \eta_p \) and the head of water supplied to the water by the pump is \( H_p \) metres of water, then the pump Horse power required is:
\[ P = \frac{\gamma_w QH_P}{75 \eta_p} \]

Since \( P(\text{in KW}) = 9.81 \, Q \, H \)

and \( P(\text{in Horse power}) = \frac{1000 \, Q \, H}{75} \), then

1 Horse power = 0.736 kW

Example 5.4

Determine the Horsepower supplied by the pump is 100 l/s of water is flowing through the system shown in Figure 5.4. The gauge reading is 100 cm and the gauge liquid is mercury, \( s = 13.6 \). What is the pump Horsepower required if its efficiency is 91.5%?

Figure E 5.4
Solution:

Since pressure at \( c = \) pressure at \( D \),

\[
P_A + (z_A - 1.0) \gamma_w + 1.0 \times 13.6 \times \gamma_w = P_B + z_B \gamma_w
\]

Dividing throughout by \( \gamma_w \),

\[
\frac{P_A}{\gamma_w} + z_A - 1.0 + 13.6 = \frac{P_B}{\gamma_w} + z_B
\]

Thus

\[
\frac{P_A}{\gamma_w} + z_A - z_B = -12.6
\]

\[
v_A = \frac{0.1 \times 4}{\pi (0.2)^2} = 3.18 \text{ m/s and } \frac{v_A^2}{2g} = \frac{3.18^2}{2 \times 9.81} = 0.52 \pi
\]

\[
v_B = \frac{0.1 \times 4}{\pi (0.15)^2} = 5.65 \text{ m/s and } \frac{v_B^2}{2g} = \frac{5.65^2}{2 \times 9.81} = 1.63 \text{ m}
\]

Let the head supplied by the pump to the system = \( h_p \). Applying the energy equation between \( A \) and \( B \) (neglecting losses) and taking the plane through \( CD \) as datum:

\[
z_A + \frac{P_A}{\gamma_w} + \frac{v_A^2}{2g} + h_p = z_B + \frac{P_B}{\gamma_w} + \frac{v_B^2}{2g}
\]

Thus

\[
h_p = \frac{P_B - P_A}{\gamma_w} + z_A - z_B + \frac{v_B^2}{2g} - \frac{v_A^2}{2g}
\]

\[
= - \left[ \left( \frac{P_A - P_B}{\gamma_w} \right) + z_B - z_A \right] + \frac{v_B^2}{2g} - \frac{v_A^2}{2g}
\]

\[
= 12.6 + 1.63 - 0.52 = 13.71 \text{ m of water}
\]
.: Horsepower supplied by the pump =

\[ = \frac{\gamma_w Q h_p}{75} \]

\[ = \frac{1000 \times 0.1 \times 13.71}{75} = 18.3 \]

The pump Horsepower required = \[\frac{\gamma_w Q h_p}{75 \times \eta_p} = \frac{18.3}{0.915} = 20\]

Example 5.5

A flow of 450 l/s of water enters a turbine through a 60 cm diameter pipe under a pressure of 147.1 kN/m². The water leaves the turbine through a 90 cm diameter pipe under a pressure of 34.32 kN/m². If a vertical distance of 2.0 m separates the centre lines of the two pipes, how much power is supplied to the turbine? If the turbine is 90% efficient, how much power is made available to the generator?

Solution:

Referring to Figure E 5.5

\[ A_1 = \frac{\pi}{4} (0.6)^2 = 0.283 \text{ m}^2, \quad v_1 = \frac{Q}{A_1} = \frac{0.45}{0.283} = 1.59 \text{ m/s} \]

\[ A_2 = \frac{\pi}{4} (0.9)^2 = 0.636 \text{ m}^2, \quad v_2 = \frac{Q}{A_2} = \frac{0.45}{0.636} = 0.708 \text{ m/s} \]

Applying the energy equation (neglecting losses) between sections (1) and (2):
Power supplied to the turbine = 9.81 x 0.45 x 13.594 = 60.01kW

\[ \frac{60.01}{0.736} = 81.54 \text{ Horsepower} \]

Power made available to the generator = 60.01 x 0.90 = 54.1 kW

\[ \frac{54.1}{0.736} = 73.38 \text{ Horsepower} \]
5.7 Piezometric Head and Total Head

Consider Bernoulli's equation i.e.

\[ z + \frac{p}{\gamma} + \frac{v^2}{2g} = \text{constant} \]

Each term in the above equation represents energy per unit weight of fluid and has the dimension of length. The sum of the elevation head and pressure head i.e. \((z + p/\gamma)\) is called piezometric head and the sum of all the three is called total head, where the elevation head \(Z\) is measured with respect to an arbitrary datum. A piezometer is a simple device used to measure positive pressures of liquids. It consists of a glass tube connected to the pipe wall in which the liquid can rise freely without overflowing. If piezometers were to be installed at different sections of the conduit shown in Figure 5.4, the liquid will rise to different levels above the centre line of the conduit.

![Diagram of Bernoulli's equation](image)

**Figure 5.4** Schematic representation of Bernoulli's equation
Figure 5.4 shows a graphical representation of Bernoulli's equation for a frictionless flow through a streamtube. If AA is a horizontal datum, the elevation of the centerline of the tube with respect to the datum at any section is the elevation head. The vertical distance, at any section, from the tube centerline to the level up to which the liquid rises in a piezometer tube represents the pressure head at that section.

The locus of all points at a distance \( (z + p/\gamma) \) from the datum A-A is called the piezometric headline or the hydraulic gradeline. The locus of all points at a distance \( (z + \frac{P}{\gamma} + \frac{v^2}{2g}) = H \) above the datum is called the total headline or the energy gradeline. Thus at any section the total headline is always above the piezometric headline by an amount equal to the velocity head \( \frac{v^2}{2g} \). If the pressure at a section is sub atmospheric i.e. negative pressure or partial vacuum, the hydraulic grade line will be below the centreline of the cross-section by an amount equal to this negative pressure head.

In real fluid flow, there is reduction of energy along the flow due to friction, minor losses due to local changes in velocity etc. If a pump is installed in a piping system, there will be an abrupt rise in the energy gradeline by an amount equal to the head supplied by the pump. Likewise, there will be an abrupt drop in the energy grade line at a turbine by an amount equal to the head extracted by the turbine. It may also be noted that both the hydraulic and energy, gradelines are straight sloping lines irrespective of the pipeline being straight or curved since the slopes of these lines are referred to per unit length of the pipe and not unit length in any specified direction.
In the following are shown typical Hydraulic gradeline (HGL) and energy gradeline \([EGL]\) sketches for various flow conditions.

5.7.1 **Gravity flow between two reservoirs through a straight pipeline.**

![Diagram](image)

**Figure 5.5** HGL and EGL for flow between two reservoirs connected by a uniform diameter pipe

For a straight and uniform diameter pipe, the EGL and HGL will be straight, parallel lines and their slope will represent the rate of head loss. There are local (minor) losses at exit from reservoir A and at entrance to reservoir B. \(H\) is the total head loss between reservoir A and B or it is the head causing the flow.
5.7.2 Pipe discharging freely into the atmosphere from a reservoir

![Diagram of HGL and EGL for pipe discharging freely into the atmosphere](image)

Figure 5.6 HGL and EGL for pipe discharging freely into the atmosphere

5.7.3 Two reservoirs connected by varying diameter pipes.

![Diagram of HGL and EGL for varying diameter pipes connecting two reservoirs](image)

Figure 5.7 HGL and EGL for varying diameter pipes connecting two reservoirs

The smaller diameter pipe in the central portion introduces a contraction at its beginning and an enlargement at its end in the direction of flow. Thus in addition to frictional losses
in the straight pipes, there will be local losses called entrance loss, contraction loss, enlargement loss, and exit loss and the HGL and the EGL will be as shown. It is seen that the HGL may rise in the direction of flow when the flow passes from a smaller to a larger diameter pipe since there will be an increase in pressure with a decrease in velocity and since the rate of energy loss and velocity head will be smaller in the larger pipe. But the EGL can never rise in the direction of flow (unless there is an external input of energy) as there is always a continuous loss of energy.

5.7.4 Free discharge through a nozzle in a pipeline containing a meter and a valve.

![Figure 5.8](image)

5.7.5 Pipeline with a pump

A pump in a pipeline system adds energy to the fluid flow and thus produces a sudden vertical rise in the hydraulic gradient.

\[ h_e = \text{entrance loss} \]
\[ h_n = \text{head loss in suction pipe} \]
Figure 5.9  HGL and EGL pipe line with a pump

\[ h_{L_d} = \text{head loss in delivery pipe} \]

\[ \frac{v_d^2}{2g} = \text{exit velocity head} = \text{exit head loss} \]

\[ h_p = \text{head supplied by the pump} \]

\[ z_s = \text{static lift} = \text{level difference between the reservoirs.} \]

It is clear from Figure 5.9 that:

\[ h_p = z_s + h_e + hl_s + hl_d + \frac{v_d^2}{2g} \]
i.e \( h_p = \text{static lift} + \text{head loss in suction pipe} + \text{head loss in delivery pipe} \).

5.7.6 Discharge through a siphon

The hydraulic gradeline for a full flow condition in a siphon will be a straight line as if the pipe were taken straight from reservoir to reservoir.

![Figure 5.10 HGL and EGL for siphone flow](image)

5.7.7 Discharge in an open channel

In open channel flow, atmospheric pressure acts on the water surface. Therefore the water surface and the hydraulic grade line should coincide.

5.7.8 Discharge over an ogee spillway

Figure 5.12 show the HGL and EGL for flow over an Ogee spillway with a hydraulic jump on the downstream apron.

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5.8 Impulse - Momentum Equation

The impulse momentum equation, along with the continuity equation and Bernoulli's Equation is the third basic tool for the solution of flow problems. Its application leads to the solution of problems in fluid mechanics involving forces and changes in velocity and which cannot be solved by the energy principle alone.
In the discussions that follow, steady one dimensional flow case will be used to develop the momentum equation since this approach has been found to be sufficient in the majority of cases.

The impulse momentum equation for fluids can be derived from Newton's 2nd Law of motion which states that the resultant external force acting on a fluid mass in any direction is equal to the time rate of change of linear momentum in that direction.

This may be obtained from Newton's Second Law of motion as follows:

\[ F = ma = m \frac{dv}{dt} \]

or

\[ F = \frac{d}{dt} (mv) = \text{rate of change of Momentum} \]

The above may be written for steady flow as:

\[ F = m \frac{dv}{dt} = \rho Q \frac{dv}{dt} = \rho Q \frac{dv}{dt} \]

\( dv \) is the change in velocity between the exit and entrance to a control volume considered.

Thus: \[ F = \rho Q (v_2 - v_1) \] (5.18)

Where \( v_2 \) is the velocity at exit and \( v_1 \) is the velocity at entrance to the control volume and \( F, v_2 \) and \( v_1 \) are in the same direction.

Consider the flow in the stream tube shown in Figure 5.13
Let AA BB be the control volume. Within the control volume the internal forces cancel out.

Summation of forces on the control volume will yield only the external forces at the control surfaces.

Forces acting on the control surfaces are:
1. Forces $F_1$ and $F_2$ at ends.
2. Weight $W$
3. Reaction force $R$ on the Whole Control Volume

Equilibrium equations in the $x$ and $y$ directions will be

$$\sum F_x = F_{1x} - F_{2x} - R_x \quad (5.19)$$

$$\sum F_y = F_{1y} - F_{2y} + R_y - W \quad (5.20)$$

Equation (5.19) and (5.20) give the net force in the $x$ and $y$ directions.
To determine the change of Momentum of the fluid as it passes through the control volumes:

Let in a small interval of time the fluid move from position AA BB to A'A' B'B'. The Mass A'A' BB does not experience any change in Momentum and may be taken as stationary. Thus the differential change in Momentum is equal to the change in Momentum as the fluid moves form AA to A'A' and BB to B'B'. Mass of fluid entering the control volume in time $dt$ is $Q_0dt$. Assuming the fluid to be incompressible, the mass of fluid leaving the control volume in the same time interval $dt$ is $Q_0dt$.

Thus the change in momentum as the fluid moves into and out of the control volume will be:

Change in Momentum = Momentum at end BB - Momentum at end AA

\[ d(Mv) = (Q_0 dt)v_2 - (Q_0 dt)v_1 \] (5.21)

The rate of change of momentum in the x and y directions may be obtained from the general equation 5.21. Thus:

\[ \frac{d}{dt} (Mv)_x = Q_0v_{2x} - Q_0v_{1x} \] (5.22)

\[ \frac{d}{dt} (Mv)_y = Q_0v_{2y} - Q_0v_{1y} \] (5.23)

The final momentum equations in the two coordinate directions may be obtained from equations 5.18, 5.22 and 5.23 as:

In the x direction:

\[ \sum F_x = \rho Q(v_{2x} - v_{1x}) \] (5.24)
In the y direction:

\[
\sum F_y = \rho \bar{Q}(v_{2y} - v_{1y})
\]  \hspace{1cm} (5.25)

For a specific situation, the expressions for \(\Sigma F_x\) and \(\Sigma F_y\) are substituted in the momentum equation from equations 5.19 and 5.20 respectively.

The velocity components in the momentum equation is assumed to be constant and is the average velocity at the cross-section considered. In situations where the velocity is not constant across a cross-section, a correction factor \(\beta\) called the momentum correction factor, similar to the kinetic energy correction factor explained in section 5.5, needs to be applied to the average velocity components. Thus for non-uniform velocity of flow,

\[
\sum F_x = \rho \bar{Q}(\beta_2 v_{x2} - \beta_1 v_{x1})
\]

\[
\sum F_y = \rho \bar{Q}(\beta_2 v_{y2} - \beta_1 v_{y1})
\]

Where \(\beta_1\) and \(\beta_2\) are the momentum correction factors at section 1 and section 2 respectively. To obtain the value of \(\beta\) at a section, the momentum based on the average velocity is equated to the integral of the momentum of the elemental stream tubes over the entire cross section.

Thus:

\[
\rho \bar{Q} \beta v_x = \int_A \rho \, dQ \cdot v_x = \rho \int_A dQ \cdot v_x
\]

where: \(dQ = v_x \, dA\) and \(Q = VA\)

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Substituting,

$$\rho V_x \cdot A \cdot \beta \cdot V_x = \rho \int_A v_x dA \cdot V_x$$

Therefore,

$$\beta = \frac{1}{V_x} \int_A \left( \frac{V_x}{V_x} \right)^2 dA$$

of more generally,

$$\beta = \frac{1}{A} \int_A \left( \frac{V}{V_x} \right)^2 dA \quad \text{(5.26)}$$

EXERCISE PROBLEMS

5.1 Water flows through a horizontal 150 mm pipe under a pressure of 4.14 bar. Assuming no losses, what is the flow if the pressure at a 75 mm diameter reduction is 1.38 bar? (Ans. $Q = 0.11$ m$^3$/s)

5.2 Water flows upwards in a vertical 300 mm pipe at the rate of 0.22 m$^3$/s. At point A in the pipe the pressure is 2.1 bar. At B, 4.6 m above A, the diameter is 600 mm and the head loss from A to B equals 1.8 m of water. Determine the pressure at B.

5.3 The pressure inside the pipe at S (fig. P 5.3) must not fall below 0.24 bar absolute. Neglecting losses, how high above water level A may point S be located? (Ans. 6.6 m)
5.4 In Fig. P 5.4 the flow was found to be inadequate and it was decided to install a pump near the tank to increase the flow by 25%. Neglecting losses calculate the required horsepower of the pump.

![Diagram](Fig. P 5.4)

5.5 A turbine is supplied with water from a reservoir which is 200 m above the level of the discharge pipe. The discharge through the pipe is 0.20 m$^3$/s. If the power output from the shaft of the turbine is 310 kW and it has a mechanical efficiency of 90 per cent, calculate (a) the power drawn from the reservoir, (b) the hydraulic power delivered to the turbine. (Ans. 392.4 kW, 344.4 kW)
5.6 A closed tank contains water with air above it. The air is maintained at a pressure of 103 kPa and 5 m below the water surface a nozzle discharges into the atmosphere. At what velocity will water emerge from the nozzle?

5.7 How much power must be supplied for the pump in Fig. P 5.7 to maintain readings of 250 mm of Mercury vacuum and 275 kPa on gages (1) and (2) respectively, while delivering a flowrate of 0.15 m³/s of water? (Ans. 54.36 kW)

![Fig. P 5.7](image)

5.8 Calculate the discharge per unit width through the frictionless sluice gate, shown in Fig. P 5.8, when the depth h is 1.5 m. Also calculate the depth h for a flow rate of 3.25 m³/s/m.

![Fig. P 5.8](image)
5.9 The turbine in Fig P 5.9 develops 75 kW when the flow rate is 0.6 m$^3$/s. What flow rate may be expected if the turbine is removed? (An. 1.118 m$^3$/s)

Fig. P 5.9

5.10 Determine the shaft horsepower for an 80 percent efficient pump to discharge 30 ℓ/s through the system of Fig P 5.10. The system losses, exclusive of pump losses, are 12 V$^2$/2g, and H = 16 m.

Fig. P 5.10
CHAPTER 6
APPLICATIONS OF BERNOULLI'S AND
MOMENTUM EQUATION

6.1 Applications of Bernoulli's Equation

6.1.1 Introduction:

Bernoulli's equation is one of the important tools for solving many problems in fluid mechanics. It is applied either singly or in combination with the continuity and momentum equations depending upon the desired result. However, the following assumptions that were made in the derivation of Bernoulli's equation should be carefully borne in mind while applying the equation,

i) The flow is assumed to be steady i.e. there is no variation in the pressure, velocity and the density of the fluid at any point with respect to time. However, Bernoulli's equation can be applied without appreciable error in problems of unsteady flow with gradually changing conditions. Thus a problem of emptying a large reservoir, where the liquid level does not drop too rapidly, can be solved by applying Bernoulli's equation inspite of the fact that the flow is strictly unsteady.

ii) Bernoulli's equation holds true strictly only along a streamline since it is derived by integrating Euler's equation of motion along a streamline. However, in fully turbulent flows where variations in velocity across a section is not appreciable, the use of the mean velocity enables the application of Bernoulli's equation without appreciable error.

iii) The flow is assumed to be incompressible. Since liquids are generally considered incompressible, Bernoulli's equation is applicable for liquids. However, the equation can
also be applied to gas flow problem when there is little variation in pressure and temperature.

iv) The equation is derived for ideal fluid where loss of energy due to friction does not exist. For real fluid flow in which frictional head loss occurs, this loss must be considered and included in Bernoulli's equation. But when the two sections considered are close to each other, frictional losses may be neglected.

Applications of Bernoulli's equation in some important devices in both closed conduit and open channel flows will be discussed in the sections that follow. In all cases loss of energy occurring is ignored in the derivation of the equations and then the theoretical results are corrected by experimentally determined coefficients to allow for the ignored loss of head.

6.1.2 The Pitot Tube

The pitot tube is used to measure the velocity of a stream. It consists of a simple L-shaped tube facing the oncoming flow (Fig. 6.1(a)). If \( u \) is the velocity of the stream at A, a particle moving from A to B will be brought to rest so that \( u \) at B is zero. B is called the stagnation point.

From Bernoulli's equation between A and B, datum through AB,

\[
\text{Total energy per unit weight at A} = \text{Total energy per unit weight at B}
\]
Thus \( \frac{P_B}{\gamma_w} = \frac{P_A}{\gamma_w} + \frac{u^2}{2g} \), since \( u_o = 0 \)

Since \( \frac{P_A}{\gamma_w} = Z \) and \( \frac{P_B}{\gamma_w} = h + Z \)

\[ \frac{u^2}{2g} = \frac{P_B - P_A}{\gamma_w} = h \]

velocity at \( A = u\sqrt{2gh} \)

---

**Figure 6.1** Pitot tube
When the Pitot tube is used in a channel, the value of \( h \) can be determined directly, as in Fig. 6.1(a). But if it is to be used in a pipe, the difference between the static pressure \( (p_s) \) and the pressure at the impact hole (i.e. the stagnation pressure \( p_b \)) must be measured with a differential pressure gauge, using a static pressure tapping in the pipe wall [Fig 6.1(b)] or a combined Pitot static tube [Fig 6.1(c)]. In the Pitot static tube, the inner tube is used to measure the impact (stagnation) pressure while the outer sheath has holes in its surface to measure the static pressure.

The theoretical velocity \( u = \sqrt{2gh} \) requires calibration to obtain the real velocity.

The real velocity \( u_r = C\sqrt{2gh} \), where \( C \) is the Pitot-tube Coefficient and \( h \) is the difference of head measured in terms of the flowing fluid. \( C \) usually varies between 0.95 and 1.00. For the Pitot-static tube [Fig. 6.1(c)] the value of \( C \) is unity for Reynolds number \( \rho uD/\mu > 3000 \), where \( D \) is the diameter of the tip of the tube.

**Example 6.1**

A pitot-static tube used to measure air velocity along a wind tunnel is coupled to a water manometer which shows a difference of head of 5 mm of water. The density of air is 1.2 kg/m\(^3\). Determine the air velocity assuming the pitot-tube coefficient is unity.
Solution:

\[ \text{Velocity of air} = C \sqrt{2gh} \]

where \( C = \text{tube coefficient} = 1.00 \)
\[ h = \text{the difference in head expressed in terms of head of the flowing fluid i.e. air.} \]
\[ = 0.005 \left( \frac{\rho_v}{\rho_s} \right) = 0.005 \left( \frac{1000}{1.2} \right) = 4.167 \text{ m of air} \]

\[ \therefore \text{Velocity of air} = 1.0 \sqrt{2 \times 9.81 \times 4.167} = 9.04 \text{ m/s} \]

6.1.3 The Venturi Meter

The Venturi meter is a device used to measure the rate of flow or the discharge \( Q \) in a pipe. It consists of a short converging conical tube leading to a cylindrical and straight portion called the "throat" which is followed by a diverging section, Fig. 6.2. The entrance and exit diameter is the same as that of the pipe line into which it is inserted. The size of a venturi meter is specified by the pipe and the throat diameter; for instance a 6 by 4 cm venturi meter fits a 6 cm diameter pipe and has a throat diameter of 4 cm. The angle of the convergent cone is between 20° and 40°, the length of the throat is equal to the throat diameter, and the angle of the divergent cone is 7° to 15°. Pressure tappings are taken at the entrance and at the throat and the pressure difference is measured by a suitable gauge. The pressure difference created as a result of the constriction is dependent on the rate of flow through the meter.

The expression for the discharge is obtained by considering sections 1 and 2 at the inlet and throat respectively.
Neglecting losses between inlet and throat and applying Bernoulli's equation between sections 1 and 2, datum through the centerline gives:

\[
z_1 + \frac{p_1}{\gamma} + \frac{v_1^2}{2g} = z_2 + \frac{p_2}{\gamma} + \frac{v_2^2}{2g}
\]

For a horizontal meter, \( z_1 = z_2 \).

Thus:

\[
\frac{v_2^2 - v_1^2}{2g} = \frac{p_1 - p_2}{\gamma}
\]  

(6.2)
From continuity equation: \( a_1 v_1 = a_2 v_2 \)

\[ \therefore v_2 = \frac{a_1}{a_2} v_1 \]

Substituting in equation 6.2,

\[ v_1^2 \left( \frac{a_1^2}{a_2^2} - 1 \right) = 2g \left( \frac{P_1 - P_2}{\gamma} \right) \]

\[ \therefore v_1 = \frac{a_2}{\sqrt{a_1^2 - a_2^2}} \cdot \sqrt{2g \frac{P_1 - P_2}{\gamma}} \]

\[ \text{Discharge } Q = a_1 v_1 = \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \cdot \sqrt{2gH} \quad (6.3) \]

where \( H = \frac{(P_1 - P_2)}{\gamma} \) = pressure difference expressed as head of the flowing fluid with specific weight \( \gamma \).

In equation 6.3, \( Q \) is the theoretical discharge.

If \( m = \frac{a_1}{a_2} \) is substituted, equation 6.3 becomes:

\[ Q = a_1 \sqrt{\frac{2gH}{(m^2 - 1)}} \]
The actual discharge is obtained by multiplying the theoretical discharge Q by the coefficient of discharge $C_d$ obtained experimentally.

Thus, Actual discharge = $C_dQ = C_d a_1 \sqrt{\frac{2gH}{m^2 - 1}}$ \hspace{1cm} (6.4)

The value of $C_d$ is a function of the ratio of throat to inlet diameter and the Reynolds number. For low diameter ratios and high Reynolds number, $C_d$ is between 0.97 and 0.99. For the setup shown in Figure 6.2, the value of the pressure difference $H$ is obtained by writing the manometric equation starting from the entry section 1 as:

$$p_1 + h_1 \cdot \gamma - x \cdot \gamma_g - (h_1 - x) \gamma = p_2$$

$$\frac{p_1 - p_2}{\gamma} = x \left( \frac{\gamma_g}{\gamma} - 1 \right)$$

It can be shown that where the pressure difference is measured by a U-tube differential manometer, the value of $x$ is independent of the inclination of the Venturi-meter.

**Example 6.2**

Water flows upwards in a 200 mm by 100 mm vertical Venturi-meter. The U-tube manometer with a gauge liquid of specific gravity 1.25 connected to the entrance and the throat registers a difference of 1 m of the gauge liquid. Taking the coefficient of the meter to be 0.99, determine the rate of flow.
Solution:

From the given data:

\[ d_1 = 0.2 \, m, \quad d_2 = 0.1 \, m, \quad x = 1 \, m, \quad S_g = 1.25 \]

and \( C_d = 0.99 \)

Thus:

\[ a_1 = \frac{n \times 0.2^2}{4} = 0.0314 \, m^2 \]

\[ a_2 = \frac{n \times 0.1^2}{4} = 0.00785 \, m^2 \]

\[ H = x \left( \frac{Y_g}{Y_w} - 1 \right) = 1 \left( \frac{1.25 \cdot Y_w}{Y_w} - 1 \right) \]

\[ = 0.25 \, m \, \text{of water} \]

\[ m = \frac{a_1}{a_2} = \frac{d_1^2}{d_2^2} = 4.0 \]

Using Equation 6.4,

\[ Q = C_d \cdot a_1 \sqrt{\frac{2gH}{m^2 - 1}} \]

\[ = 0.99 \times 0.0314 \sqrt{\frac{2 \times 9.81 \times 0.25}{4 \times 4 - 1}} \]

\[ = 0.0178 \, m^3/s = 17.8 \, l/s \]
6.1.4 The Orifice Meter

The orifice meter consists of a concentric, sharp-edged circular orifice made in a thin plate which is clamped between the flanges of a pipe, Figure 6.3. It is used for measuring the flow in a pipe, by relating the pressure difference between a section immediately upstream of the plate (section 1) and the vena Contracta of the issuing jet downstream of the plate (section 2).

Figure 6.3 shows an orifice plate inserted in a pipeline. The fluid passing through the orifice contracts in area. The section of the stream where the cross-sectional area is minimum is called the Vena Contracta and forms at a distance of about \(d_1/2\) downstream from the plane of the plate, where \(d_1\) is the pipe diameter.
The flow cross-sectional area at the vena contracta is minimum and the velocity is maximum and hence the pressure is minimum. Thus, as in the case of the venturi meter, the discharge may be calculated by measuring the pressure difference between the centers of the sections (1) and (2).

Applying Bernoulli's equation between the centers of the sections (1) and (2), datum through the centre line of the pipe:

\[
\frac{p_1}{\gamma} + \frac{v_1^2}{2g} = \frac{p_2}{\gamma} + \frac{v_2^2}{2g}
\]

or \[
\frac{v_2^2 - v_1^2}{2g} = \frac{p_1 - p_2}{\gamma} = H
\]

From continuity equation, \(a_1v_1 = a_2v_2\)

\[v_1 = \frac{a_2}{a_1} \cdot v_2 = c_c \cdot \frac{a_o}{a_1} \cdot v_2, \quad \text{where} \quad c_c = \frac{a_2}{a_o}
\]

Thus: \[
\frac{v_2^2}{2g} \left[ 1 - c_c^2 \cdot \frac{a_o^2}{a_1^2} \right] = h
\]

and \(v_2 = \left[ \frac{2gH}{1 - c_c^2 \cdot \frac{a_o^2}{a_1^2}} \right]^{\frac{1}{2}}\)

Introducing the coefficient of velocity \(c_v\), the actual velocity \(v_{2a}\):

\[v_{2a} = c_v \left[ \frac{2gH}{1 - c_c^2 \cdot \frac{a_o^2}{a_1^2}} \right]^{\frac{1}{2}}\]
The actual discharge \( Q \) will be

\[
Q = a_2 \cdot v_{2a} = c_c \cdot a_o \cdot v_{2a}
\]

or

\[
Q = c_d \cdot a_o \left[ \frac{2gH}{1 - c_c^2 \frac{a_o^2}{a_1^2}} \right]^{1/2}
\]

where \( c_c a_o = a_2 \) and \( c_c c_v = C_d \)

The above equation for \( Q \) may further be simplified by absorbing the two coefficients and the other constants into a single coefficient to give:

\[
Q = c A_o \sqrt{2gH}
\]

If the differential manometer reads a gauge difference \( x \), it can be shown that:

\[
H = \frac{p_1 - p_2}{\gamma} = x \left( \frac{\gamma_g}{\gamma} - 1 \right)
\]

\[
\therefore Q = c A_o \cdot \sqrt{2gx \left( \frac{\gamma_g}{\gamma} - 1 \right)} \quad (6.5)
\]

The orifice coefficient \( C \) depends upon the ratios of the orifice and pipe areas and the Reynolds number of the flow. The coefficient \( C \) of the orifice meter is much lower than that of the venturi meter with values normally ranging from 0.6 to 0.65.
Example 6.3

An orifice meter, fixed to a 25 cm diameter pipe, has a diameter of 10 cm. The pipe conveys oil of specific gravity 0.9. Calculate the discharge if a mercury differential manometer reads a difference of 80 cm and \( C = 0.65 \).

Solution:

From equation 6.5

\[
Q = CA_o \cdot \sqrt{2gx \left( \frac{\gamma_g}{\gamma} - 1 \right)}
\]

\[
= 0.65 \times \frac{\pi}{4} (0.10)^2 \sqrt{2 \times 9.81 \times 0.8 \left( \frac{13.6}{1} - 1 \right)}
\]

Then \( Q = 0.0718 \text{ m}^3/\text{s} = 71.8 \text{ l/s} \)

6.1.4 Flow Through an Orifice

An orifice is an opening in the side or bottom of a tank or reservoir through which liquid is discharged in the form of a jet, normally into the atmosphere. Normally orifices are circular in cross-section. The flow velocity and thus the discharge through an orifice depend upon the head of the liquid above the level of the orifice. The flow is thus strictly speaking unsteady since the flow velocity varies with varying head as the outflow continues. However, for large tanks where the drop in level is small compared to the velocity of outflow through the orifice, steady flow may be assumed and Bernoulli's equation applied without appreciable error. Distinction will be made between small orifice and large orifice.
Small Orifice: The term 'small orifice' is used for an orifice which has diameter or vertical dimension small compared to the head producing the flow so that the head is assumed not to vary appreciably from point to point across the orifice. Figure 6.4 shows a small opening in the side of a large tank containing a liquid with specific weight $\gamma$ and with a free surface open to the atmosphere. At point A, the pressure $P_A$ is atmospheric and the velocity $V_A$, i.e., the rate of drop of the reservoir level will be negligibly small if the tank is large. Point B is a point in the vena contracta where $P_B$ is atmospheric and the velocity $V_B$ is equal to the velocity $V$ of the jet.

![Diagram of flow through a small orifice](image)

**Figure 6.4 Flow through a small orifice**

Applying Bernoulli's equation between A and B, datum through the center of the orifice and neglecting loss of energy between A and B:

$$Z_A + \frac{P_A}{\gamma} + \frac{V_A^2}{2g} = Z_B + \frac{P_B}{\gamma} + \frac{V_B^2}{2g}$$

$$H + 0 + 0 = 0 + 0 + \frac{V^2}{2g}$$

(6.6)

... velocity of jet $V = \sqrt{2gH}$
Equation 6.6 is a statement of Torricelli's theorem, which is that the velocity of the issuing jet is proportional to the square root of the head producing the flow. If A is the cross-sectional area of the opening, then

Theoretical discharge \( Q = A \sqrt{2gH} \)

The actual discharge through the orifice is much less than the theoretical discharge and must be obtained by introducing a discharge coefficient \( C_d \), so that

\[
Q_{\text{actual}} = C_d \cdot Q = C_d \cdot A \sqrt{2gH}
\]  
(6.7)

The actual discharge is less than the theoretical discharge because:

i) The actual velocity of the jet is less than that given by equation 6.6 since there is a loss of energy between A and B. Thus

\[
\text{actual velocity at } B = c_v \cdot v = c_v \cdot \sqrt{2gH}
\]  
(6.8)

where \( c_v \) is coefficient of velocity, which has to be determined experimentally. \( c_v \) is of the order of 0.97.

ii) The area of the issuing jet at the vena contracta i.e. at B, is less than the area of the orifice opening

\[
\text{Actual area of jet at } B = c_c A
\]  
(6.9)
where \( c_e \) is coefficient of contraction and depends on the profile of the orifice. For sharp edged orifice \( c_e \) is of the order of 0.64.

Thus, from equation 6.8 and 6.9, the actual discharge will be:

\[
\text{Actual discharge} = \text{Actual velocity at B} \times \text{Actual area at B} = c_v \cdot \sqrt{2gH} \cdot c_e \cdot A = c_v c_e A \sqrt{2gH}
\]

Comparison of the above with equation 6.7 shows that:

\[
s_{d} = c_{c} c_{v}
\]

(6.10)

To determine the discharge coefficient \( s_d \), the actual volume passing through the orifice in a given time is collected and compared with the theoretical discharge.

Then,

\[
s_{d} = \frac{\text{Actual measured discharge}}{\text{Theoretical discharge}}
\]

Similarly, by measuring the actual area of the jet and the velocity at the vena contracta, the coefficients of contraction \( C_c \) and coefficient of velocity \( C_v \) could be determined as:

\[
C_c = \frac{\text{Area of jet at vena contracta}}{\text{Area of orifice}}
\]

\[
C_v = \frac{\text{Velocity at vena contract}}{\text{Theoretical velocity}}
\]

For the case where the orifice is in the side of a tank (i.e. not at the bottom), measurement of the profile of the jet enables the determination of the actual velocity of the jet and thus that of \( C_v \). Referring to Figure 6.5:
Figure 6.5 Profile of a jet

In Figure 6.5, point B is the vena contracta and at point C the jet falls a distance $y$ vertically in a horizontal distance $x$ from the vena contracta. Let $t$ be the time taken for a fluid particle to travel from B to C. If air resistance is negligible and the horizontal component of the velocity $v$ remains unchanged, then the distance travelled in time $t$ will be:

$$x = v \cdot t$$

and since the initial vertical velocity component at B is zero, the vertical distance $y$ travelled in the same time $t$ will be:

$$y = \frac{1}{2} g t^2$$

from which: \( v = \frac{x}{t} \) and \( t = \sqrt{\frac{2y}{g}} \)
so that \( \nu = \sqrt{\frac{gx^2}{2y}} \) = actual velocity of jet at B.

Since theoretical velocity at B = \( \sqrt{2gH} \), then

\[
C_v = \frac{\text{Actual velocity}}{\text{Theoretical velocity}} = \frac{\sqrt{g x^2 / 2y}}{\sqrt{2gH}} = \sqrt{x^2/4yH}
\]

Hydraulic coefficients of some typical orifices and mouth pieces are given in figure 6.6 below.

![Hydraulic coefficients for some typical orifices and mouthpieces](image_url)

Figure 6.6 Hydraulic coefficients for some typical orifices and mouthpieces

**Example 6.4**

Find the diameter of a circular orifice to discharge 0.015 m\(^3\)/s under a head of 2.4 m using a coefficient of discharge of 0.6.

If the orifice is in a vertical plane and the jet falls 0.25 m in a horizontal distance of 1.3 m from the vena contracta, find the value of the coefficient of contraction.
Solution:

From the given data:

\[ Q = 0.015 \text{ m}^3/\text{s}, \quad H = 2.4 \text{ m} \quad C_d = 0.6 \]
and \( y = 0.25, \ x = 1.3 \text{ m}. \)

\[ Q = C_d A \sqrt{2gH} = C_d \cdot \frac{d^2}{4} \sqrt{2gH} \]

\[ \therefore d = \left( \frac{4Q}{C_d \pi \sqrt{2gH}} \right)^{\frac{1}{2}} = \left( \frac{4 \times 0.015}{0.6 \pi \sqrt{19.62 \times 2.4}} \right)^{\frac{1}{2}} \]

\[ = 0.0681 \text{ m} = 6.81 \text{ cm} \]

i.e diameter of the orifice = 6.81 cm

\[ C_v = \sqrt{\frac{X^2}{4yH}} = \sqrt{\frac{1.3^2}{4 \times 0.25 \times 2.4}} = 0.8391 \]

Since \( c_d = c_c C_v \),

\[ C_c = \frac{C_d}{C_v} = \frac{0.6}{0.8391} = 0.715 \]

Large Orifice: An orifice is classified as 'large' when the vertical height of the orifice is large so that the head producing the flow is substantially less at the top of the opening then at the bottom. The discharge calculated using the formula of small orifice, where the head \( H \) is measured to the centre of the orifice, will not be the true value since the velocity the will vary substantially from top to bottom of the opening. In this case theoretical discharge is calculated by
integrating from top to bottom the flow through thin horizontal strips across the orifice.

Consider the large rectangular orifice of width $B$ and depth $D$ shown in Figure 6.7.

![Figure 6.7 Flow through a large orifice](image)

As shown in Figure 6.6 the top and bottom of the orifice opening are at depth $H_1$ and $H_2$ respectively below the free surface.

Consider a horizontal strip across the opening of height $dh$ at a depth $h$ below the free surface.

Area of the strip = $Bdh$

Velocity of flow through the strip = $\sqrt{2gh}$

Discharge through strip, $dQ = B\sqrt{2gh} \cdot h^{\frac{1}{2}} dh$

To obtain the discharge through the whole opening, integrate $dQ$ from $h = H_1$ to $h = H_2$. 

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\[ \text{Discharge } Q_t = B \sqrt{2g} \int_{h_1}^{h_2} h^{\frac{1}{3}} \, dh = \frac{2}{3} B \sqrt{2g} \left( H_2^{3/2} - H_1^{3/2} \right) \]

The actual discharge \( Q = \frac{2}{3} C_d B \sqrt{2g} \left[ H_2^{3/2} - H_1^{3/2} \right] \).

Example 6.5

Water flows from a reservoir through a rectangular opening 2 m high and 1.2 m wide in the vertical face of a dam. Calculate the discharge in m³/s when the free surface in the reservoir is 0.5 m above the top of the opening assuming a coefficient of discharge of 0.64.

Solution

Referring to Figure 6.6:

\[ D = 2 \text{ m}, \quad B = 1.2 \text{ m}, \quad H_1 = 0.5 \text{ m} \quad C_d = 0.64 \]

\[ \therefore \quad H_2 = H_1 + D = 0.5 + 2 = 2.5 \text{ m.} \]

\[ \therefore \quad Q = \frac{2}{3} C_d B \sqrt{2g} \left[ H_2^{3/2} - H_1^{3/2} \right] \]

\[ = \frac{2}{3} \times 0.64 \times 1.2 \times \sqrt{19.62} \left[ 2.5^{3/2} - 0.5^{3/2} \right] \]

\[ = 2.2679 \times (3.9528 - 0.3536) \]

\[ = 8.16 \text{ m}^3/\text{s} \]

Thus, the discharge is 8.16 m³/s
Unsteady Flow Through Small Orifice: Problem of discharge through an orifice under varying head strictly fall under unsteady flow. But if the rate of fall of the head is very small compared to the velocity of efflux, Bernoulli's equation may be conveniently applied without appreciable error. The following two cases of unsteady flow through small orifice are of practical importance and will be considered:

i) Time required for a desired fall of liquid level in a tank due to efflux from an orifice.

ii) Flow from one tank to another.

Time required to empty a tank of uniform cross-section:
Consider a tank of uniform cross-sectional area $A$ discharging liquid through an orifice of cross-sectional area $a$ installed at its bottom as shown in Fig. 6.8.

![Figure 6.8 Flow through a small orifice at tank bottom](image)

Let the height of the liquid be at $h$ above the vena contracta at some instant. The theoretical outflow velocity at that instant will be:

$$v = \sqrt{2gh}$$
Let the liquid level fall by an amount \( dh \) during a time interval \( dt \). The volume of liquid that has flown out in time \( dt \) will be:

\[
dV = -Adh
\]

Volume of liquid that has passed through the orifice in the same time interval \( dt \) will be:

\[
dV = C_d a \sqrt{2gh} \cdot dt
\]

Thus:

\[
C_d a \sqrt{2gh} \cdot dt = -Adh
\]

\[
: \quad dt = \frac{-Adh}{C_d a \sqrt{2gh}} = \frac{-A(h^{\frac{1}{2}})}{C_d a \sqrt{2gh}} dh
\]

The time \( T \) required for the liquid level to drop from \( H_1 \) to \( H_2 \) may be found by integrating the above equation between the limits \( H_1 \) and \( H_2 \).

\[
T = \int dt = \frac{-A}{C_d a \sqrt{2g}} \int_{H_1}^{H_2} h^{-\frac{1}{2}} dh
\]

\[
i.e \quad T = \frac{-2A}{C_d a \sqrt{2g}} (H_2^{1/2} - H_1^{1/2})
\]

Since \( H_1 > H_2 \), the term in brackets is negative, thus \( T \) will be positive. Taking the minus sign out of the bracket;

\[
T = \frac{2A}{C_d a \sqrt{2g}} (H_1^{1/2} - H_2^{1/2}) \quad (6.11)
\]
The tank will be fully emptied when $H_2 = 0$.

Equation 6.11 gives the required time in seconds.

**Flow from one tank to another through an orifice:**

Consider two adjacent tanks of uniform cross-sectional area $A_1$ and $A_2$ connected by an orifice of cross-sectional area $a$ as shown in Figure 6.9.

![Figure 6.9 Flow between adjacent vessels through an orifice](image)

Let $H_1 = \text{initial difference between the liquid levels in the two tanks}$

$H_2 = \text{final difference in level between the liquid levels in the two tanks}$

At any instant, let the difference in levels be $H$.

The theoretical velocity of the liquid through the orifice at this instant is

$$ v = \sqrt{2gH} $$

After a small time interval $dt$, let the fall in head in tank $A_1$ be $dh$. The volume that has gone out of tank $A_1$ will be $dhA_1$. If $y$ is the change in level of tank $A_2$, then the volume coming into tank $A_2$ in time $dt$ will be $yA_2$. 

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From continuity, \( y \cdot A_2 = dhA_1 \).

\[ \therefore y = dh \cdot \frac{A_1}{A_2} \]

Then the total change in head difference between \( A_1 \) and \( A_2 \) will be

\[ dH = dh + y = dh \left( 1 + \frac{A_1}{A_2} \right) \]

Equating the flow through the orifice for the time \( dt \) to the volume of displacement:

\[ -A_1 \, dh = C_d a \sqrt{2gH} \cdot dt \]

or

\[ dt = \frac{-A_1 \, dh}{C_d a \sqrt{2gH}} = \frac{-A_1 \, dH}{C_d a \sqrt{2gH} \left( 1 + \frac{A_1}{A_2} \right)} \]

Integrating the above between \( H_1 \) and \( H_2 \), the time \( T \) required for the level difference in the two tanks to drop from \( H_1 \) to \( H_2 \) is

\[ T = \frac{2A_1 \left( H_1^{1/2} - H_2^{1/2} \right)}{C_d a \sqrt{2g} \left( 1 + \frac{A_1}{A_2} \right)} \quad (6.12) \]

The time required for the level between the two tanks to equalize is obtained when \( H_2 = 0 \).
Example 6.6

A rectangular tank 10 m x 6 m has an orifice with 10 cm diameter fitted at its bottom. It water stands initially at a height of 5 m above the orifice, what time is required for the level to drop to 1 m above the orifice. Take the orifice coefficient to be 0.64.

Solution:

The time required for the level to drop from $H_1$ to $H_2$ due to flow through an orifice fitted at the bottom of a tank is given by Equation 6.11 as:

$$T = \frac{2A(H_1^{1/2} - H_2^{1/2})}{C_d a \sqrt{2g}}$$

Here; $A = 10 \times 6 = 60 \, M^2$

$H_1 = 5 \, m, \quad H_2 = 1 \, m$

$$a = \frac{\pi d^2}{4} = \frac{\pi}{4} (0.1)^2 = 0.00785 \, m^2$$

$$T = \frac{2 \times 60 \left(5^{1/2} - 1^{1/2}\right)}{0.64 \times 0.00785 \sqrt{2} \times 9.81} = 6665.37 \, sec$$

i.e time required is: 1.851 hrs.
Example 6.7

Two tanks having plan areas of 6 m × 3 m and 1.5 m × 2 m are connected by a circular orifice 20 cm in diameter. Before the flow through the orifice began, the difference in water levels in the tanks was 4 m, with the higher level in the larger tank. Determine the time required to bring the difference down to 1.2 m. Take \( C_d = 0.61 \).

Solution:

From equation 6.12, the time required to bring the level difference between the two tanks from \( H_1 \) to \( H_2 \) is given by

\[
T = \frac{2A_1(H_1^{1/2} - H_2^{1/2})}{C_d a \sqrt{2g} \left( 1 + \frac{A_1}{A_2} \right)}
\]

where, \( A_1 = \text{area of larger tank} = 6 \times 2 = 12 \text{ m}^2 \)
\( A_2 = \text{area of smaller tank} = 1.5 \times 2 = 3 \text{ m}^2 \)
\( a = \text{area of orifice} = \frac{\pi}{4} (0.2)^2 = 0.0314 \text{ m}^2 \)

\( H_1 = 4 \text{ m} \)
\( H_2 = 1.2 \text{ m} \)
\( C_d = 0.61 \)

\[
T = \frac{2 \times 12(4^{1/2} - 1.2^{1/2})}{0.61 \times 0.0314 \sqrt{2} \times 9.81 \left( 1 + \frac{12}{3} \right)} = 51.18 \text{ sec}
\]
6.1.6 Notches and Weirs

A notch is a geometrical opening in the side of a tank or reservoir extending above the free surface or it may be defined as any regular obstruction in open stream over which the flow takes place. It is in effect a large orifice which has no upper edge so that it has a variable flow area depending on the level of the free surface. A weir is a notch on a large scale, used, for example, to measure the discharge of a stream or a river.

A notch or a weir may be classified according to

a) Shape of the opening: as rectangular, triangular, trapezoidal etc.

b) Shape of the edge: as sharp-crested and broad-crested.

c) Discharge condition: as free or submerged.

d) End condition: as weir with end contractions and weir without end contraction i.e suppressed weir.

End Conditions: If the length of the crest of the notch or weir is equal to the width of the approach channel, then there will be no end contractions of the stream at the sides and the width of the nappe or jet flowing over the crest will be equal to the length of the crest. This type of weir in which end contractions are suppressed is called suppressed weir (Figure 6.10). Figures 6.10(b) and 6.10(c) illustrate weirs with one and two end contractions respectively.
Discharge Over Sharp Crested Rectangular Weir:

The sharp-crested rectangular weir, shown in Figure 6.11, has a sharp-edged horizontal crest which is normal to the flow. The nape falling over the crest is contracted at top and bottom as shown.

An equation for the discharge over the sharp-crested rectangular weir can be derived by neglecting the contractions of the nape. Without contractions, the flow appears as shown in Figure 6.12 with the nape having parallel streamlines with
atmospheric pressure throughout.

Figure 6.12 Weir nappe without contraction

Neglecting the approach velocity and losses and applying Bernoulli's equation between (1) and (2), datum through the crest,

\[ H + 0 + 0 = \frac{v^2}{2g} + (H - y) + 0 \]

Solving for \( v \),

\[ v = \sqrt{2gy} \]

The theoretical discharge \( Q_t \) is:

\[ Q_t = \int v \, dA = \int_0^H v \, B \, dy = \sqrt{2g} \, B \int_0^H y^{1/2} \, dy \]

or \[ Q_t = \frac{2}{3} \sqrt{2g} \, B \, H^{3/2} \]

where \( B = \) length of the weir crest.
The actual discharge $Q_a$ is obtained by introducing a discharge coefficient $C_d$ to the theoretical discharge. Thus:

$$Q_a = C_d Q_t$$

$$Q_a = \frac{2}{3} C_d \sqrt{2g} B H^{3/2}$$

The discharge coefficient $C_d$ is a function of $H$ and $P$ and can be estimated from:

i) Bazin's formula:

$$C_d = \left( 0.607 + \frac{0.00451}{H} \right) \left[ 1 + 0.55 \left( \frac{H}{P + H} \right)^2 \right]$$

where $H = \text{head over crest in metres}$

$P = \text{height of crest above channel floor in metres}$

ii) Rehbock formula:

$$C_d = 0.605 + \frac{1}{1048H - 3} + \frac{0.08H}{P}$$

where $H$ and $P$ are in metres.

This formula is valid for a notch with no end contractions.

Generally, however, $C_d = 0.62$ where by:

$$Q_a = 1.84 BH^{3/2}$$

(6.13)
Rectangular Weir with end Contraction:

When the weir crest does not extend completely across the full width of the channel it is said to have end contractions as shown in Figure 6.13. In this case the effective width, $B_e$, of the crest is less than $B$ as a result of the end contraction. Francis found that the end contraction for each contraction is about $0.1\,H$, where $H$ is the head over the weir crest.

![Rectangular weir with end contraction](image)

Thus $B_e = (B - 0.1nH)$, where $n = \text{number of end contractions}$.

For a fully contracted rectangular weir, $n = 2$

$$\therefore B_e = B - 0.2H$$

Thus $Q_a = 1.84(B - 0.2H)H^{3/2}$

When the height $P$ of a weir is small, the approach velocity head at point (1) cannot be neglected. In such a situation, a correction may be added to the head as:

$$Q = \frac{2}{3} C_d B \sqrt{2g \left( H + \alpha \frac{v^2}{2g} \right)}$$

where $v = \frac{Q}{B(P + H)}$ and $\alpha = 1.4$
The above equation must be solved for \( Q \) by trial since both \( v \) and \( Q \) are unknown. As a first trial the term \( \alpha \frac{v^2}{2g} \) may be neglected to approximate \( Q \). With this trial discharge, a value of \( v \) is computed.

**The V-notch or Triangular Weir:**

The V-notch or triangular weir, shown in Figure 6.14, is particularly convenient for measuring small discharges. The contraction of the nappe is neglected and the discharge is computed as follows:

![Figure 6.14 V-notch or Triangular weir](image)

The velocity at depth \( y \) in Figure 6.14 is given by

\[
\nu = \sqrt{2gy}
\]

The theoretical discharge \( Q \), is

\[
Q_t = \int v \, dA = \int_y^H v \, x \, dy
\]
From similar triangles, \( x \) may be related to \( y \):

\[
\frac{x}{H - y} = \frac{L}{H}
\]

\[
\therefore \quad Q_t = \sqrt{2g} \cdot \frac{L}{H} \int_0^H (H - y) \frac{dy}{y^{1/2}} = \frac{4}{15} \sqrt{2g} \cdot \frac{L}{H} H^{5/2}
\]

But \( \frac{L}{2H} = \tan \frac{\phi}{2} \)

\[
\therefore \quad Q_t = \frac{8}{15} \sqrt{2g} \tan \frac{\phi}{2} H^{5/2}
\]

The actual discharge is obtained by introducing a discharge coefficient \( C_d \). Thus:

\[
Q_a = \frac{8}{15} C_d \sqrt{2g} \tan \frac{\phi}{2} H^{5/2}
\]

(6.14)

where \( C_d = f\left(\frac{H}{P}, \frac{P}{B}, \phi\right) \)

for \( \phi = 90^\circ \), \( C_d = 0.58 \) so that

\[
Q_a = 1.38 H^{2.5}
\]

(6.15)

**Trapezoidal Weir:** A Trapezoidal Weir, shown in Figure 6.15, can be considered to be made up of a rectangular weir of width \( B \) and a triangular weir of apex angle \( \phi \).

If \( C_d \) and \( C_d' \) represent the discharge coefficients of the rectangular weir and the triangular weir respectively, the
Figure 6.15 Trapezoidal weir

discharge \( Q \) flowing through the trapezoidal weir under head \( H \) will be

\[
Q = \frac{2}{3} C_d \sqrt{2g} \cdot B \cdot H^{3/2} + \frac{8}{15} C'_d \sqrt{2g} \tan \frac{\phi}{2} \cdot H^{5/2}
\]  

(6.16)

Since the rectangular weir is contracted, the discharge will be reduced by \( \frac{2}{3} C_d \cdot \sqrt{2g} \cdot (0.2H) \cdot H^{3/2} \). If this reduction in discharge due to contraction is thought of as being compensated by increase in discharge due to the triangular portion, then

\[
\frac{2}{3} C_d \cdot \sqrt{2g} \cdot (0.2H) \cdot H^{3/2} = \frac{8}{15} C'_d \sqrt{2g} \cdot \tan \frac{\phi}{2} \cdot H^{5/2}
\]

from which one obtains \( \tan \frac{\phi}{2} = \frac{1}{4} \)

This particular trapezoidal weir in which the side slopes with 1 horizontal to 4 vertical is known as Cippoletti Weir. Thus, for a Cippoletti weir,
The Broad-Crested Weir: A broad-crested weir is one where the crest of the weir is broad i.e. the crest width $B > 0.4 \, H$ (Figure 6.16)

![Figure 6.16 Broad-crested weir](image)

The upstream edge of the weir is rounded to avoid separation. Neglecting losses and assuming a parallel stream of flow with hydrostatic pressure distribution over the crest, Bernoulli's equation applied between points (1) and (2), neglecting approach velocity, gives:

$$H + 0 + 0 = \frac{v_2^2}{2g} + z + (y - z)$$

$$\therefore \quad v_2 = \sqrt{2g(H - y)}$$

For a weir of width $L$ normal to the plane of the figure, the theoretical discharge is

$$Q_t = v_2 \cdot L \cdot y = Ly \sqrt{2g(H - y)}$$
For \( y = H, Q = 0 \). Maximum \( Q \) occurs for a particular value of \( y \). This value is obtained by differentiating \( Q \) with respect to \( y \) and equating the result to zero for maximum \( Q \). Thus

\[
\frac{dQ}{dy} = 0 = L\sqrt{2g(H - y)} + Ly \cdot \frac{1}{2} \cdot \frac{-2g}{\sqrt{2g(H - y)}}
\]

\[
\therefore [2g(H - y)] = L \cdot y \cdot g
\]

from which \( y = \frac{2}{3} H = y_c \)

Thus : \( V_2 = \sqrt{2g \left( \frac{3}{2} y - y \right)} = \sqrt{g} y \)

and \( Q_t = 1.705 LH^{3/2} \)

The actual discharge is obtained by introducing a discharge coefficient \( C_d \). Thus

\[
Q_a = 1.705 C_d LH^{3/2}
\]

For a well-rounded upstream edge, \( C_d = 0.98 \).

\[
\therefore Q_a = 1.67 LH^{3/2} \quad (6.18)
\]

Example 6.8

The discharge over a suppressed rectangular weir is to be 0.20 m\(^3\)/s when the head over the crest is 30 cm. If the discharge coefficient is 0.6, calculate the length of weir crest required.
Solution:

\[ Q = \frac{2}{3} C_d \cdot B \sqrt{2g} \cdot H^{3/2} \]

Substituting \( Q = 0.20 \text{ m}^3/\text{s}, \ C_d = 0.6 \) and \( H = 0.30 \text{ m}, \)

\[ B = \frac{\frac{2}{3} Q}{C_d \sqrt{2g} \cdot H^{3/2}} = \frac{0.20}{\frac{2}{3} \times 0.6 \times \sqrt{19.62} \cdot (0.3)^{3/2}} = 0.69 \text{ m} \]

Example 6.9

Determine the discharge over a sharp-crested rectangular weir with 8 m crest length and a head of 2.4 m. The width of the approach channel is 10 m. Take \( C_d = 0.622. \)

Solution:

This is a rectangular weir with end contractions. Thus, neglecting the approach velocity,

\[ Q = \frac{2}{3} C_d (B - 0.2H) \sqrt{2g} \cdot H^{3/2} \]

Substituting \( C_d = 0.622, \ B = 8 \text{ m}, \ H = 2.4 \text{ m}, \)

\[ Q = \frac{2}{3} \times 0.622 (8 - 0.2 \times 2.4) \sqrt{2g} \cdot 2.4^{3/2} \]

\[ = 51.36 \text{ m}^3/\text{s} \]

Example 6.10

A 90° V-notch has a discharge coefficient of 0.60. Calculate the discharge when the observed head is 0.65 m.
Solution:

For a V-notch, the discharge is given by:

\[ Q = C_d \cdot \frac{8}{15} \sqrt{2g} \cdot \tan \frac{\phi}{2} \cdot H^{5/2} \]

Substituting \( C_d = 0.60, \ H = 0.65 \text{ m} \) and \( \phi = 90^\circ \),

\[ Q = 0.60 \times \frac{8}{15} \sqrt{19.62} \tan 45^\circ \cdot (0.65)^{5/2} \]

\[ = 0.483 \text{ m}^3/\text{s} \]

6.2 Applications of the Momentum Equation

6.2.1 Introduction

The momentum equation is used in the solution of the following two classes of problems:

i) To determine the resultant force acting on the boundary of a flow passage by a stream of fluid as the stream changes its direction or magnitude or both. Problems of this type are forces on pipe bends, reducers, stationary and moving vanes, jet propulsion etc.

ii) To determine the characteristics of flow when there is an abrupt change of flow section such as in sudden enlargement in a pipe and hydraulic jump in channels and also to determine the forces involved in structures in open channel flow.

Typical cases of the two types of problems will be discussed in the sections that follow.
6.2.2 Dynamic force due to a jet impinging on a stationary surface.

Force on a flat plate:

Consider a stationary, smooth plate on which a jet of cross-sectional area 'a' impinges with a velocity \( V_0 \) inclined at an angle \( \theta \) with the plate. Let \( \rho \) be the density of the fluid and assume the plate and the jet to be in a horizontal plane. Assume also no frictional and impact losses at the plate so that the velocity \( V_0 \) remains unchanged.

When a jet strikes a solid surface a stream of fluid is formed which moves over the surface and it leaves the surface tangentially.

For the control volume shown,

Continuity equation gives: \( Q_0 = Q_1 + Q_2 \) \hspace{1cm} (a)

In the y direction no force is exerted by the plate on the fluid. Thus:
\[ \sum F_y = 0 = \rho Q_1 V_o - \rho Q_2 V_o - \rho Q_o V_o \cos \theta = 0 \]

from which: \[ Q_1 - Q_2 = Q_o \cos \theta \]  

(b)

Solving (a) and (b)

\[ Q_1 = \frac{Q_o}{2} (1 + \cos \theta) \]

\[ Q_2 = \frac{Q_o}{2} (1 - \cos \theta) \]

In the x-direction, the plate exerts a force \( F \) on the fluid in the control volume in a direction normal to the plate as shown. Thus:

\[ \sum F_x = -R = \rho Q_o \times 0 - \rho Q_o V_o \sin \theta \]

\[ \therefore R = \rho Q_o V_o \sin \theta \]

(6.19)

Hence the jet exerts an equal and opposite force to \( R \) on the plate in the positive x-direction.

If the inclined plate moves with a velocity \( v \) say in the same direction as the jet, then the mass of fluid impinging on the plate per unit time will be \( \rho A (V_o - v) \) and it will be less than the mass impinging on a stationary plate.

Summing forces in the x-direction:

\[ -R = \rho A (V_o - v) [0 - (V_o - v) \sin \theta] \]

\[ = -\rho A (V_o - v)^2 \sin \theta \]

or \[ R = \rho A (V_o - v)^2 \sin \theta \]

(6.20)
**Force on a curved vane:**

**Fixed vanes:**

Since momentum is a vector quantity, change in momentum would occur across a control volume when there is change in direction only, with or without change in the magnitude of velocity. Consider the flow of jet of area $A$ of fluid of density $\rho$ impinging on a fixed curved vane shown in Figure 6.18 with a constant velocity $V_o$.

![Figure 6.18 Force on a fixed curved vane](image)

The entrance and exit angles of the curved vane are $\alpha$ and $\beta$ with respect to the $x$-axis.

The Momentum of the fluid at inlet in the $x$ direction is $\rho A V \cos \alpha$ and at exit it is $-\rho A V_o \cos \beta$. Hence,

$$\sum F_x = -F = -\rho A V_o^2 \cos \beta - \rho A V_o^2 \cos \beta$$

$$\therefore F = \rho A V_o^2 (\cos \alpha + \cos \beta) \quad (6.21)$$
The force $F$ shown in the figure is the force exerted by the vane on the fluid. An equal and opposite force is exerted by the fluid on the vane. The force will be maximum when $\alpha = \beta = 0$ i.e. when the vane is semi-circular.

**Moving Vane:**

Consider a curved vane moving at a velocity $v$ in the $x$-direction as shown in Fig. 6.19.

![Figure 6.19 Force on a moving curved vane](image)

A jet of absolute velocity $V_i (= A0)$ of a fluid density $\rho$ and area $A$ is directed at the vane at an angle $\alpha_i$ to the $x$ direction i.e. the direction of motion of the vane. The jet will enter the vane with a velocity $B_0$, which is the relative velocity of the absolute velocity $V_i$ with respect to the vane velocity $v (= AB)$. $B_0$ is called relative velocity $V_{r_i}$. This relative velocity $V_{r_i}r1$ of the jet needs to be tangential to the inlet blade angle $\beta_1$ for the jet to enter the vane smoothly. For smooth vane surface, the jet moves along the vane without change in the magnitude of the relative velocity. Let it, however, be assumed that there is some change in velocity and
the velocity of the jet becomes $V_{r2}$ as it emerges from the vane with the vane outlet angle $\beta_2$. The absolute velocity of the jet leaving the vane will be $V_2$. $V_{r2}$ is the relative velocity of $V_2$ with respect to the vane velocity $v$. $V_2$ is at an angle $\alpha_2$ with the $x$-axis.

Considering the control volume (shown in dashed lines, Figure (6.19)) of the fluid enclosed by the inlet, the vane and the outlet, the mass of water entering the control volume per unit time is $\rho AV_{r1}$. Applying the momentum equation in the $x$ direction:

$$\sum F_x = -F_x = \rho AV_{r1}(-V_2 \cos \alpha_2 - V_1 \cos \alpha_1)$$

which reduces to:

$$F_x = \rho AV_{r1} (V_1 \cos \alpha_1 + V_2 \cos \alpha_2) \quad (6.22)$$

The force $F_x$ shown in Figure 6.19 is the force exerted by the vane on the fluid.

Similarly,

$$\sum F_y = F_y = \rho AV_{r1} (V_2 \sin \alpha_2 - V_1 \sin \alpha_1)$$

$$\therefore F_y = \rho AV_{r1} (V_2 \sin \alpha_2 - V_1 \sin \alpha_1) \quad (6.23)$$

The resultant force $R$ acting on the fluid by the vane is:

$$R = \sqrt{F_x^2 + F_y^2}$$

and \(\tan \phi = \frac{F_y}{F_x}\)
An equal and opposite force to the force R shown in Figure 6.19 will be exerted by the water on the vane.

**Example 6.11**

A jet of water 4.0 cm in diameter and having a velocity of 20.0 m/s impinges on a flat plate normally. Find the force exerted on the plate if

a) The plate is stationary
b) The plate is moving with velocity of 3.0 m/s in the same direction and what is the work done?

**Solution:**

a) Mass of water striking the plate when the plate is stationary is:

\[ \rho AV = 1000 \times \frac{\pi}{4} \times (0.04)^2 \times 20 = 25.13 \text{ kg/s} \]

Force exerted on the plate is:

\[ F = M(V - 0) = 25.13 \times 20 = 502.6 \text{ N} \]

b) When the plate is moving in the same direction as the jet, the mass of water striking the plate will be

\[ \rho A(V - v) = 1000 \times \frac{\pi}{4} \times (0.04)^2 \times (20 - 3.0) = 21.36 \text{ kg/s} \]

Force exerted on the plate will be:

\[ F = M(V - v) = 21.36 \times 17 = 363.13 \text{ N} \]

Work done = \( F \cdot v = 363.13 \times 3 = 1.089.39 \text{ W} \)

\[ = 1.089 \text{ kW} \]
Example 6.12

A 5.0 cm diameter jet of water having a velocity of 40 m/s strikes a vane (see Figure E) having a deflection angle of 135° and moving at a velocity of 15 m/s in the same direction. Assuming no friction, compute:

i) The x and y components of the force exerted by the fluid on the vane.

ii) Absolute velocity of the jet when it leaves the vane and

iii) The power developed.

![Figure E 6.12]

Solution:

i) The relative velocity of the jet with respect to the vane is (40-15) = 25 m/s

Therefore, discharge striking the vane will be:

\[
\frac{\pi}{4} (0.05)^2 \times 25 = 0.0491 \text{ m}^3/\text{s}
\]
Since there is no friction the relative velocity as the jet leaves the vane will also be 25 m/s
Thus:

\[
\sum F_x = \rho Q [-25 \cos 45 - 25]
\]
\[
= 1000 \times 0.0491 [-17.68 - 25] = -2095.59 N
\]
\[
\sum F_y = \rho Q [V_x \cos 45 - 0]
\]
\[
= 1000 \times 0.0491 [25 \cos 45 - 0] = 868.09 N
\]

Therefore, the force components exerted by the fluid on the vane will be:

\[F_x = +2095.59 N\]
\[F_y = -868.09 N\]

ii) The absolute velocity of the jet when it leaves the vane at exit can be obtained by combining the relative velocity with vane velocity vectorially as shown by the velocity triangle.

![Velocity Triangle Diagram]

AB = relative velocity at exit = 25 m/s
BC = vane velocity = 15 m/s
AC = Absolute velocity of jet at exit.

Since \(AD = AB \sin 45^\circ = 25 \times \sin 45^\circ = 17.68 \text{ m/s}\)
and \(CD = BD - BC = AB \cos 45 - 15 = 17.68 - 15 = 2.68 \text{ m/s}\)
Then \( AC = \sqrt{17.68^2 + 2.68^2} = \sqrt{319.76} = 17.88 \text{ m/s} \)

i.e. The absolute velocity as the jet leaves the vane is 17.88 m/s

iii) Power developed will be

\[ F_x \cdot v = \frac{2095.59 \times 15}{1000} = 31.434 \text{ kW} \]

Note that no work is done by the force \( F_y \) since the vane is not moving in that direction.

6.2.2 Dynamic Force due to Flow Around a Bend

Flow in a pipe bend, in a vertical or horizontal plane, and with or without change in diameter, experiences change in momentum. As a result of this change in momentum, a dynamic force is exerted by the fluid on the bend which has to be resisted by a thrust block or other suitable means. The force could be evaluated by a simple application of the momentum equation to the fluid mass in the control volume between the entrance and exit of the bend.

Consider a reducing bend in a vertical plane shown in Figure 6.20 (a)

Figure 6.20 (b) shows the control volume and the forces acting on the fluid mass within the control volume. Assuming steady flow \( Q \) of a fluid of density \( \rho \) the momentum equation applied in the \( x \) and \( y \) directions gives the force components \( R_x \) and \( R_y \) exerted by the bed on the fluid as follows:
In the x-direction:

$$\sum F_x = P_1A_1 - P_2A_2\cos\theta - R_x = \rho Q(v_2\cos\theta - v_1)$$

$$\therefore R_x = P_1A_1 - P_2A_2\cos\theta - \rho Q(v_2\cos\theta - v_1)$$

In the y-direction:

$$\sum F_y = R_y - \dot{W} - P_2A_2\sin\theta = \rho Q(v_2\sin\theta - 0)$$

$$\therefore R_y = \rho Qv_2\sin\theta + \dot{W} + P_2A_2\sin\theta$$

Thus the resultant force $R$ will be

$$R = \sqrt{R_x^2 + R_y^2}$$

and

$$\phi = \tan^{-1} \frac{R_y}{R_x}$$
However, for a horizontal bend, the weight $W$ of the fluid mass between sections (1) and (2) of the bend will drop out of the momentum equation in the $y$-direction.

**Example 6.13**

The following data are given for a 60° reducer bend in a horizontal plane shown in Figure E 6.13. $D_1 = 15$ cm, $D_2 = 10$ cm, $Q = 0.106 \text{ m}^3/\text{s}$ and $P_1 = 205.94 \text{ kN/m}^2$. Assuming no loss as water flows from section (1) to section (2), determine the force required to hold the bend in place.

![Figure E 6.13](image)

**Solution:**

From the given data:

$$V_1 = \frac{Q}{A_1} = \frac{0.106 \times 4}{\pi \times (0.15)^2} = 5.998 \text{ m/s}; \quad \frac{v_1^2}{2g} = 1.834 \pi$$

$$V_2 = \frac{Q}{A_2} = \frac{0.106 \times 4}{\pi (0.1)^2} = 13.496 \text{ m/s}; \quad \frac{v_2^2}{2g} = 9.284 \pi$$
To determine the pressure at section (2) apply Bernoulli's
equation between sections (1) and (2):

\[
\frac{P_1}{\gamma_w} + \frac{v_1^2}{2g} + Z_1 = \frac{P_2}{\gamma_w} + \frac{v_2^2}{2g} + Z_2
\]

\[
\frac{205.94}{9.81} + 1.834 + 0 = \frac{P_2}{\gamma_w} + 9.284 + 0
\]

\[
\therefore \frac{P_2}{\gamma_w} = 20.993 + 1.834 - 9.284 = 13.543 \text{ m of water}
\]

\[
\therefore P_2 = 13.543 \times \gamma_w = 132.857 \text{ kN/m}^2
\]

Applying the Momentum equation in the coordinate directions;

x-direction:

\[
\sum F_x = P_1A_1 - P_2A_2\cos60^\circ - F_x = \rho Q(v_2\cos60^\circ - v_1)
\]

\[
i.e \ F_x = P_1A_1 - P_2A_2\cos60^\circ - \rho Q(v_2\cos60^\circ - v_1)
\]

\[
P_1A_1 = 205.94 \times 10^3 \times \frac{\pi}{4} (0.15)^2 = 3639.26 \text{ N}
\]

\[
P_2A_2 = 132.857 \times 10^3 \times \frac{\pi}{4} (0.1)^2 = 1043.46 \text{ N}
\]

\[
\therefore F_x = 3639.26 - 1043.46 \cos60^\circ
\]

\[
- 1000 \times 0.106 (13.496 \cos60^\circ - 5.998)
\]

\[
= 3639.26 - 521.73 - 79.5 = 3038.03 \text{ N}
\]
y-direction:

\[ \sum F_y = F_y - P_2 A_2 \sin 60^\circ = \rho Q (v_2 \sin 60^\circ - 0) \]

\[ \therefore F_y = P_2 A_2 \sin 60^\circ + \rho Q (v_2 \sin 60^\circ) \]

\[ = 1043.46 \sin 60^\circ + 1000 \times 0.106 (13.496 \sin 60^\circ) \]

\[ = 903.66 + 1238.92 = 2142.58 \text{ N} \]

The force required to hold the bend in place is \( R \)

\[ R = \sqrt{3038.03^2 + 2142.58^2} = 3717.56 \text{ N} \]

\[ \phi = \tan^{-1} \frac{2142.58}{3038.03} = 35.19^\circ \]

6.2.4 Dynamic Force at a Nozzle

A nozzle, attached to a pipeline, and discharging to the atmosphere provides a good example of a rapid change in velocity. The fluid exerts a force on the nozzle and in accordance with Newton's third law there is a similar force, of opposite sign, exerted by the nozzle on the fluid. This is the force which the tension bolts holding the nozzle with the pipe must be designed to withstand.

Consider the nozzle shown in Figure 6.21 discharging a fluid of density \( \rho \) with a velocity \( v_2 \) to the atmosphere. The flow velocity at the entrance into the nozzle i.e. at section (1) is \( v_1 \).

Application of the momentum equation between upstream section (1) and downstream section (2) will yield the force \( R_x \) exerted by the nozzle on the fluid.
Figure 6.21 Nozzle discharging to atmosphere

In Figure 6.21 the component forces acting on the control volume are the pressure forces $P_1 A_1$ and $P_2 A_2$ and the force $R_x$ exerted by the nozzle on the fluid. The rate of change of momentum is $p Q (v_2 - v_1)$.

The momentum equation in the horizontal direction gives:

$$\sum F_x = P_1 A_1 - P_2 A_2 - R_x = \rho Q (v_2 - v_1)$$

Since the nozzle discharges to the atmosphere, $p_2 = 0$. Thus:

$$R_x = P_1 A_1 - \rho Q (v_2 - v_1) \quad (6.24)$$

**Example 6.14**

Calculate the tension force on the flanged connection between a 64 mm diameter pipe and a nozzle discharging a jet of water with velocity of 30 m/s and diameter of 19 mm.
Solution:

Let section (1) be the entrance and section (2) be the exit from the nozzle.

The discharge \( Q = \frac{\pi}{4}D_2^2 \cdot v_2 = \frac{\pi}{4}(0.019)^2 \times 30 = 0.0085 \text{ m}^3/\text{s} \)

From continuity, \( D_1^2 v_1 = D_2^2 v_2 \), so that:

\[
v_1 = \left( \frac{D_2}{D_1} \right)^2 \cdot v_2 = \left( \frac{19}{64} \right)^2 \times 30 = 2.65 \text{ m/s}
\]

Applying Bernoulli's equation between entry to and exit from the nozzle, and neglecting losses, one obtains:

\[
\frac{p_1}{\gamma_w} + \frac{v_1^2}{2g} = \frac{v_2^2}{2g}
\]

Thus \( p_1 = \frac{\gamma_w}{2g} (v_2^2 - v_1^2) = \frac{9.81}{2 \times 9.81} (30^2 - 2.65^2) = 4446.5 \text{ kN/m}^2 \)

Substituting in Equation 6.24,

\[
R_x = 446.5 \times 10^3 \times \frac{\pi}{4} (0.064)^2 - 10^3 \times 0.0085 (30 - 2.65)
\]

\[
= 1436.39 - 232.48 = 1203.9 \text{ N} = 1.204 \text{ kN}
\]

Thus, the tension force on the flanged connection is 1.204 kN.
6.2.5 Force Exerted on a Sluice Gate.

Water issues at relatively high velocity from the opening caused by the raising of a sluice gate such as the one shown in Figure 6.22. The flow behavior resembles that of a jet issuing from an orifice. The difference is, however, in that the presence of the bed prevents the pressure inside the issuing jet downstream from a sluice gate from becoming atmospheric throughout. The pressure distribution a short distance from the opening, i.e at section (2), may be approximated to be hydrostatic.

![Diagram of sluice gate](image)

Figure 6.22 Flow under a vertical sluice gate

Application of Bernoulli's equation between section (1) and (2) gives the discharge as:

$$Q = CA\sqrt{h_1}$$  \hspace{1cm} (6.25)

Where, C is an overall coefficient of discharge incorporating the coefficient of contraction, the effects of downstream head, velocity of approach and energy loss, and A is the area of the opening of the gate.

Since there is a change in velocity between section (1) and (2), there is a change in momentum leading to a force $F$ exerted
on the gate. The momentum equation may be conveniently applied to the control volume of fluid between sections (1) and (2).

Assuming hydrostatic pressure distribution at sections (1) and (2) and that the gate is installed in a wide rectangular channel where the discharge per unit width is \( q \), the forces acting on the control volume are:

- The hydrostatic force per unit width at section (1)
  \[ F_1 = \gamma_w \cdot \frac{h_1^2}{2} \]

- The hydrostatic force per unit width at section (2)
  \[ F_2 = \gamma_w \cdot \frac{h_2^2}{2} \]

- The force \( F \) per unit width of channel exerted by the gate on the fluid.

The change in momentum between section (1) and (2) per unit width of channel is \( \rho q(v_2 - v_1) \).

Thus:

\[
\sum F_x = \gamma_w \frac{h_1^2}{2} - \gamma_w \frac{h_2^2}{2} - F = \rho q(v_2 - v_1), \quad \text{so that:}
\]

\[
F = \frac{\gamma_w}{2} (h_1^2 - h_2^2) - \rho q(v_2 - v_1) \tag{6.26}
\]

The force exerted by the fluid on the gate is equal and opposite to the force \( F \) shown in Figure 6.22.
Example 6.15

The sluice gate in Figure 6.22 spans a wide rectangular channel and is raised 0.25 m above the channel floor. The upstream depth $h_1$ is 3 m. Estimate the horizontal force per metre width of channel exerted on the gate. Take $C = 2.5$ in Equation 6.25.

Solution:

The discharge per unit width is $q$.

$$q = 2.5 \times 0.25 \times 3^{1/2} = 1.08 \text{ m}^3/\text{s}$$

Since $q = v_1 \cdot h_1$, $v_1 = q/h_1 = 1.08/3 = 0.36 \text{ m/s}$

Applying Bernoulli's equation between (1) and (2):

$$3 + \frac{v_1^2}{2g} = h_2 + \left(\frac{1.08}{h_2}\right)^2 \frac{1}{2g}$$

from which, $h_2 = 0.14 \text{ m}$

Thus $v_2 = 1.08/0.14 = 7.71 \text{ m/s}$

Substituting in Equation 6.26

$$F = \frac{9810}{2} (3^2 - 0.14^2) - 1000 \times 1.08 (7.71 - 0.36)$$

$$F = 44048 - 7938 = 36110 \text{ N} = 36.11 \text{ kN}$$
EXERCISE PROBLEMS

6.1 A venture meter installed in a horizontal water main has a throat diameter of 75 mm and a pipe diameter of 150 mm. The coefficient of discharge is 0.97. Calculate the flow rate if the difference of level in a mercury U-tube gauge connected to the throat and full bore tappings is 178 mm, the mercury being in contact with the water.

(Ans. 0.0292 m³/s)

6.2 A 200 mm diameter water pipe has a venturi meter of throat diameter 12.5 cm, which is connected to a mercury manometer showing a gauge level difference of 87.8 cm. Find the velocity in the throat and the discharge. If the upstream pressure is 60 kN/m, what power would be given up by the water if it were allowed to discharge to atmospheric pressure?

6.3 A vertical cylindrical tank, 0.6 m in diameter and 1.5 m high, has an orifice of 25 mm diameter in the bottom. The discharge coefficient is 0.61. If the tank is originally full of water, what time is required to lower the level by 0.9 m? (Ans. 192 sec.)

6.4 Determine the equation of trajectory of a jet of water discharging horizontally from a small orifice with head of 5.0 m and a velocity coefficient of 0.96. Neglect air resistance.

6.5 Compute the discharge from the tank shown in Fig. P 6.5.

(Ans. 0.0241 m³/s)
6.6 A 90° v-notch and a rectangular weir and placed in series. The length of the rectangular weir is 0.6 m and its coefficient is 1.81. If the discharge coefficient of the v-notch is 0.61, what will be its working head when the head on the weir is 0.15 m?

6.7 A closed tank partially filled with water discharges through an orifice of 12.5 mm diameter and has a coefficient of discharge of 0.70. If air is pumped into the upper part of the tank, determine the pressure required to produce a discharge of 0.6 l/s when the water surface is 0.90 m above the outlet. (Ans. 15.7 kN/m²)

6.8 How long does it take to raise the water surface of the tank in Fig P 6.8 by 2.0 m? The left hand surface is that of a large reservoir of constant water surface elevation.
6.9 A jet of water, 6.5 cm² in cross-sectional area, moving at 12 m/s, it turned through an angle of 135° by a curved vane. The vane is moving at 4.5 m/s in the same direction as the jet. Neglecting any loss of velocity by shock or friction, find the amount of work done on the plate per sec. (Ans. 281 W)

6.10 Determine the head on a 60° v-notch for a discharge of 170 ℓ/s.

6.11 A 100 mm diameter orifice discharges 44.6 ℓ/s of water under a head of 2.75 m. A flat plate held normal to the jet just downstream from the vena contracta requires a force of 320 N to resist impact of the jet. Find $C_d$, $C_v$, and $C_c$.
(Ans. $C_d = 0.733$, $C_v = 0.977$, $C_c = 0.791$)

6.12 Calculate the magnitude and direction of the vertical and horizontal components and the total force exerted on the stationary vane, Fig. P 6.12, by a 50 mm jet of water moving at 15 m/s.
6.13 The blade shown in Fig. P 6.13 is one of a series. Calculate the force exerted by the jet on the blade system.

(Ans: 2651 N)

6.14 Calculate the magnitude and direction of the horizontal and vertical components of the force exerted by the flowing water on the 'flip bucket' AB. Assume that the water between sections A and B weighs 2.70 kN and that downstream from B, the moving fluid may be considered to be a free jet. (Fig. P 6.14).
6.15 When 300 ℓ/s of water flows through the vertical 300 mm by 200 mm pipe reducer bend, the pressure at the entrance is 70 kPa. Calculate the force by the fluid on the bend if the volume of the bend is 0.85 m³.

**Ans:**

\[ F_x = 8172 \text{ N} \]
\[ F_y = 4218 \text{ N} \]
\[ \theta = 27.3^\circ \]

6.16 The plate is Fig. P 6.16 covers the 125 mm diameter hole. What is the maximum \( H \) that can be maintained without leaking?

**Ans:**

\[ F_x = 8172 \text{ N} \]
\[ F_y = 4218 \text{ N} \]
\[ \theta = 27.3^\circ \]
6.17 For the weir shown in Fig. P 6.17, determine the magnitude and direction of the horizontal component of the force on the structure.

(Ans: 18.937 kN/m) in the downstream direction.
The jet of water of 50 mm diameter moving at 30 m/s is divided in half by a "splitter" on the stationary flat plate (Fig. P 6.18). Calculate the magnitude and direction of the force on the plate. Assume that the flow is in a horizontal plane.

Fig. P 6.18
REFERENCES


